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INSTITVTIONVM CALCVLI INTEGRALIS

VOLVMEN PRIMVM

IN QVO METHODVS INTEGRANDI A PRIMIS PRINCIPIIS VSQVE AD INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS PERTRACTATVR.

Editio altera et correctior.

PETROPOLI

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INDEX CAPITVM,

in Volumine primo contentorum.

Sectio

Pag. 304. lin. penult. lege $\partial y + yy\partial x$ loco $\partial x + yy\partial x$.

PRAE-

PRAENOTANDA.
DE
CALCVLO INTEGRALI
IN GENERE.

Definitio 1.

1.

Calculus integralis est methodus, ex data differentialium relatione inueniendi relationem ipsarum quantitatum: et operatio, qua hoc praestatur, integratio vocari solet.

Corollarium 1.

2. Cum igitur calculus differentialis ex data relatione quantitatum variabilium, relationem differentialium inuestigare doceat: calculus integralis methodum inuersam suppeditat.

Corollarium 2.

3. Quemadmodum scilicet in Analysi perpetuo binae operationes sibi opponuntur, veluti subtractio additioni, diuisio multiplicationi, extractio radicum euectioni ad potestates, ita etiam simili ratione calculus integralis calculo differentiali opponitur.

Corollarium 3.

4. Proposita relatione quacunque inter binas quantitates variabiles x et y, in calculo differentiali methodus traditur rationem differentialium $\partial y : \partial x$ inuestigandi: sin autem vicissim ex hac differentialium ratione ipsa quantitatum x et y relatio sit definienda, hoc opus calculo integrali tribuitur.

Scholion 1.

5. In calculo differentiali iam notaui, quaestionem de differentialibus non absolute sed relatiue esse intelligendam, ita vt, si y fuerit functio quaecunque ipsius x, non tam ipsum eius differentiale ∂y, quam eius ratio ad differentiale ∂x sit definienda. Cum enim omnia differentialia per se sint nihilo aequalia, quaecunque functio y fuerit ipsius x, semper est $\partial y = 0$, neque sic quicquam amplius absolute quaeri posset. Verum quaestio ita rite proponi debet, vt dum x incrementum capit infinite paruum adeoque euanescens ∂x, definiatur ratio incrementi functionis y, quod inde capiet, ad istud ∂x: etsi enim vtrumque est $= 0$, tamen ratio certa inter ea intercedit, quae in calculo differentiali proprie inuestigatur. Ita si fuerit $y = xx$, in calculo differentiali ostenditur esse $\frac{\partial y}{\partial x} = 2x$, neque hanc incrementorum rationem esse veram, nisi incrementum ∂x, ex quo ∂y nascitur, nihilo aequale statuatur. Verum tamen, hac vera differentialium notione obseruata, locutiones communes, quibus differentialia quasi absolute enunciantur, tolerari possunt, dummodo semper in mente saltem ad veritatem referantur. Recte ergo dicimus, si $y = xx$, fore $\partial y = 2x\partial x$, tam etsi falsum non esset, si quis diceret $\partial y = 3x\partial x$, vel $\partial y = 4x\partial x$, quoniam ob $\partial x = 0$ et $\partial y = 0$, hae aequalitates aeque subsisterent; sed prima sola rationi verae $\frac{\partial y}{\partial x} = 2x$ est consentanea.

Scholion 2.

6. Quemadmodum calculus differentialis apud Anglos methodus fluxionum appellatur, ita calculus integralis ab iis methodus fluxionum inuersa vocari solet, quandoquidem a fluxionibus ad quantitates fluentes reuertitur. Quas enim nos quantitates variabiles vocamus, eas Angli nomine magis idoneo quantitates fluentes vocant, et earum incrementa infinite parua seu euanescentia fluxiones nominant, ita vt fluxiones ipsis idem

idem ſint, quod nobis differentialia. Haec diuerſitas loquendi ita iam vſu inualuit, vt conciliatio vix vnquam ſit expectanda; equidem Anglos in formulis loquendi lubenter imitarer, ſed ſigna quibus nos vtimur, illorum ſignis longe anteferenda videntur. Verum cum tot iam libri vtraque ratione conſcripti prodierint, huiusmodi conciliatio nullum vſum eſſet habitura.

Definitio 2.

7. Cum functionis cuiuscunque ipſius x differentiale huiusmodi habeat formam $X \partial x$, propoſita tali forma differentiali $X \partial x$, in qua X ſit functio quaecunque ipſius x, illa functio, cuius differentiale eſt $= X \partial x$, huius vocatur integrale, et praefixo ſigno $\int$ indicari ſolet: ita vt $\int X \partial x$ eam denotet quantitatem variabilem, cuius differentiale eſt $= X \partial x$.

Corollarium 1.

8. Quemadmodum ergo propoſitae formulae differentialis $X \partial x$ integrale, ſeu ea functio ipſius x, cuius differentiale eſt $= X \partial x$, quae hac ſcriptura $\int X \partial x$ indicatur, inueſtigari debeat, in calculo integrali eſt explicandum.

Corollarium 2.

9. Vti ergo littera ∂ ſignum eſt differentiationis, ita littera $\int$ pro ſigno integrationis vtimur, ſicque haec duo ſigna ſibi mutuo opponuntur, et quaſi ſe deſtruunt: ſcilicet $\int \partial X$ erit $= X$, quia ea quantitas denotatur cuius differentiale eſt ∂X, quae vtique eſt X.

Corollarium 3.

10. Cum igitur harum ipſius x functionum

$$x^2,\ x^n,\ \sqrt{(aa - xx)}$$

differentialia ſint

$$2x\partial x,\ nx^{n-1}\partial x,\ \frac{-x\partial x}{\sqrt{(aa-xx)}}$$

 ſigno

figno integrationis $\int$ adhibendo, patet fore:

$$\int 2x\partial x = xx;\ \int nx^{n-1}\partial x = x^n;\ \int\frac{-x\partial x}{\sqrt{(aa-xx)}} = \sqrt{(aa-xx)}$$

vnde vſus huius ſigni clarius perſpicitur.

Scholion 1.

11. Hic vnica tantum quantitas variabilis in computum ingredi videtur, cum tamen ſtatuamus tam in calculo differentiali quam integrali, ſemper rationem duorum pluriumue differentialium ſpectari. Verum etſi hic vna tantum quantitas variabilis x apparet, tamen reuera duae conſiderantur; altera enim eſt ipſa illa functio, cuius differentiale ſumimus eſſe $X\partial x$, quae ſi deſignetur littera y, erit $\partial y = X\partial x$, ſeu $\frac{\partial y}{\partial x} = X$, ita vt hic omnino ratio differentialium $\partial y : \partial x$ proponatur, quae eſt $= X$, indeque erit $y = \int X\partial x$: hoc autem integrale non tam ex ipſo differentiali $X\partial x$, quod vtique eſt $= 0$, quam ex eius ratione ad ∂x inueniri eſt cenſendum. Caeterum hoc ſignum $\int$ vocabulo *ſummae* efferri ſolet, quod ex conceptu parum idoneo, quo integrale tanquam ſumma omnium differentialium ſpectatur, eſt natum; neque maiore iure admitti poteſt, quam vulgo lineae ex punctis conſtare concipi ſolent.

Scholion 2.

12. At calculus integralis multo latius quam ad huiusmodi formulas integrandas patet, quae vnicam variabilem complectuntur. Quemadmodum enim hic functio vnius variabilis x ex data differentialis forma inueſtigatur; ita calculus integralis quoque extendi debet ad functiones duarum pluriumue variabilium inueſtigandas, cum relatio quaedam differentialium fuerit propoſita. Deinde calculus integralis non ſolum ad differentialia primi ordinis adſtringitur, ſed etiam praecepta tradere debet, quorum ope functiones tam vnius quam duarum pluriumue variabilium inueſtigari queant, cum relatio quaedam differen-

ferentialium fecundi altiorisue cuiusdam ordinis fuerit data. Atque hanc ob rem definitionem calculi integralis ita instruximus, vt omnes huiusmodi inuestigationes in se complecteretur; differentialia enim cuiusque ordinis intelligi debent, et voce relationis, quae inter ea proponatur, sum vsus, vt latius pateret voce rationis, quae tantum duorum differentialium comparationem indicare videatur. Ex his ergo diuisionem calculi integralis constituere poterimus.

Definitio 3.

13. Calculus integralis diuiditur in duas partes, quarum prior tradit methodum, functionem vnius variabilis inueniendi ex data quadam relatione inter eius differentialia tam primi quam altiorum ordinum.

Pars autem altera methodum continet, functionem duarum pluriumue variabilium inueniendi, cum relatio inter eius differentialia siue primi siue altioris cuiusdam gradus fuerit proposita.

Corollarium 1.

14. Prout ergo functio ex data differentialium relatione inuenienda, vel vnicam variabilem complectitur, vel duas pluresue, inde calculus integralis commode in duas partes principales dispescitur, quibus exponendis duos libros destinamus.

Corollarium 2.

15. Semper igitur calculus integralis in inventione functionum vel vnius vel plurium variabilium versatur, cum scilicet relatio quaepiam inter eius differentialia siue altioris cuiuspiam ordinis fuerit proposita.

Scholion.

16. Cum hic primam partem calculi integralis in investigatione functionum vnicae variabilis ex data differentialium

relatione constituamus, plures partes pro numero variabilium functionem ingredientium constitui debere videatur, ita vt pars secunda functiones duarum variabilium, tertia trium, quarta quatuor etc. complectatur. Verum pro his posterioribus partibus methodus fere eadem requiritur, ita vt si inuentio functionum duas variabiles inuoluentium fuerit in potestate, via ad eas, quae plures variabiles implicant, satis sit patefacta; vnde inuentionem eiusmodi functionum, quae duas pluresue variabiles continent, commode coniungimus, indeque vnicam partem calculi integralis constituimus, posteriori libro tractandam.

Caeterum haec altera pars in elementis adhuc nusquam est tractata, etiamsi eius vsus in Mechanica ac praecipue in doctrina fluidorum maximi sit vsus. Quocirca cum in hoc genere praeter prima rudimenta vix quicquam sit exploratum, noster secundus liber de calculo integrali admodum erit sterilis, ac praeter commemorationem eorum, quae adhuc desiderantur, parum erit expectandum; verum hoc ipsum ad scientiae incrementum multum conferre videtur.

Definitio 4.

17. Vterque de calculo integrali liber commode subdiuiditur in partes pro gradu differentialium, ex quorum relatione functionem quaesitam inuestigari oportet. Ita prima pars versatur in relatione differentialium primi gradus, secunda in relatione differentialium secundi gradus, quorsum etiam differentialia altiorum graduum ob tenuitatem eorum, quae adhuc sunt inuestigata, referri possunt.

Corollarium 1.

18. Vterque ergo liber constabit duabus partibus, in quarum priore relatio inter differentialia primi gradus proposita considerabitur, in posteriore vero eiusmodi integrationes occur-

current, vbi relatio inter differentialia ſecundi altiorumue graduum proponitur.

Corollarium 2.

19. In primi ergo libri parte prima eiusmodi functio variabilis x inuenienda proponitur, vt poſita ea functione $= y$, et $\frac{\partial y}{\partial x} = p$, relatio quaecunque data inter has tres quantitates x, y et p adimpleatur: ſeu propoſita quacunque aequatione inter has ternas quantitates, vt indoles functionis y ſeu aequatio inter x et y tantum, excluſa p, eruatur.

Corollarium 3.

20. Poſterioris autem partis primi libri quaeſtiones ita erunt comparatae, vt poſito $\frac{\partial y}{\partial x} = p$, $\frac{\partial p}{\partial x} = q$, $\frac{\partial q}{\partial x} = r$ etc. ſi proponatur aequatio quaecunque inter quantitates x, y, p, q, r etc. indoles functionis y per x, ſeu aequatio inter x et y eliciatur.

Scholion 1.

21. Quae adhuc in calculo integrali ſunt elaborata maximam partem ad libri primi partem primam ſunt referenda, in qua excolenda Geometrae imprimis operam ſuam collocarunt: pauca ſunt quae in parte poſteriore ſunt praeſtita, et alter liber, quem ſecundum fecimus, etiamnunc fere vacuus eſt relictus. Prima autem pars libri primi, in qua potiſſimum noſtra tractatio conſumetur, denuo in plures ſectiones diſtinguitur, pro modo relationis, quae inter quantitates x, y et $p = \frac{\partial y}{\partial x}$ proponitur. Relatio enim prae caeteris ſimpliciſſima eſt, quando $p = \frac{\partial y}{\partial x}$ aequatur functioni cuipiam ipſius x, qua poſita $= X$, vt ſit $\frac{\partial y}{\partial x} = X$ ſeu $\partial y = X \partial x$; totum negotium in integratione formulae differentialis $X \partial x$ abſoluitur: huius operationis iam ſupra mentionem fecimus, quae vulgo ſub titulo integrationis formularum differentialium ſimplicium, ſeu vnicam varia-

variabilem inuoluentium tractari solet. Eodem res rediret, si $p = \frac{\partial y}{\partial x}$ aequaretur functioni ipsius y tantum, quandoquidem quantitates x et y ita inter se reciprocantur, vt altera tanquam functio alterius spectari possit: haec ergo ad sectionem primam referentur. Sin autem $p = \frac{\partial y}{\partial x}$ aequaetur expressioni ambas quantitates x et y inuoluenti, aequatio habetur differentialis huius formae $P\partial x + Q\partial y = 0$, vbi P et Q sunt expressiones quaecunque ex x, y et constantibus conflatae. Quanquam autem Geometrae multum in huiusmodi aequationum integratione desudarunt, tamen vix vltra quosdam casus satis particulares sunt progressi. Sin autem p magis complicate par x et y determinatur, vt eius valor explicite exhiberi nequeat, veluti si fuerit:

$$p^3 = xxp^3 - xyp + x^5 - y^5$$

ne via quidem constat tentanda, quomodo inde relatio inter x et y inuestigari queat: pauca ergo, quae hic tradere licebit, cum praecedentibus secundam sectionem primae partis libri primi occupabunt. Ita ex vniuersa nostra tractatione magis patebit, quod adhuc in calculo integrali desideretur, quam quid iam sit expeditum, cum hoc prae illo vt minima quaedam particula sit spectandum.

Scholion 2.

22. In singulis partibus, quas enarrauimus, fieri etiam solet, vt non solum vna quaedam functio, sed etiam simul plures inuestigentur, ita vt neutra sine reliquis definiri possit, quemadmodum in Algebra communi vsu venit, vt ad solutionem problematis plures incognitae in calculum sint introducendae, quae deinceps per totidem aequationes determinentur. Veluti si eiusmodi binae functiones y et z ipsius x sint inueniendae, vt sit:

$$x\partial y + azz\partial x = 0, \text{ et } xx\partial z + bxy\partial y = c\partial y:$$

hinc nouae subdiuisiones nostrae tractationis constitui possent.

Verum

Verum quia hic vt in Algebra communi totum negotium ad eliminationem vnius litterae revocatur, vt deinceps duae tantum variabiles in vna aequatione fuperfint, hinc tractatio non multiplicanda videtur.

Scholion 3.

23. In fecundo libro calculi integralis, quo functio duarum pluriumue variabilium ex data differentialium relatione inveftigatur, multo maior quaeftionum varietas locum habet. Sit enim z functio binarum variabilium x et t inueftiganda, et cum $(\frac{\partial z}{\partial x})$ denotet rationem eius differentialis ad ∂x, fi fola x pro variabili habeatur, at $(\frac{\partial z}{\partial t})$ rationem eius differentialis ad ∂t, fi fola t variabilis fumatur; prima pars eiusmodi continebit quaeftiones, in quibus certa quaedam relatio inter quantitates x, t, z et $(\frac{\partial z}{\partial x})$, $(\frac{\partial z}{\partial t})$ proponitur, et quaeftio huc redit, vt hinc aequatio inter folas quantitates x, t et z eruatur; inde enim qualis z fit functio ipfarum x et t, patebit. In fecunda parte praeter has formulas $(\frac{\partial z}{\partial x})$ et $(\frac{\partial z}{\partial t})$ etiam iftae $(\frac{\partial \partial z}{\partial x \partial x})$, $(\frac{\partial \partial z}{\partial x \partial t})$ et $(\frac{\partial \partial z}{\partial t \partial t})$, in computum ingredientur: quarum fignificatio ita eft intelligenda, vt pofitis prioribus $(\frac{\partial z}{\partial x}) = p$ et $(\frac{\partial z}{\partial t}) = q$, vbi p et q iterum certae erunt functiones ipforum x et t, futurum fit fimili expreffionis modo,

$$(\frac{\partial \partial z}{\partial x \partial x}) = (\frac{\partial p}{\partial x}); (\frac{\partial \partial z}{\partial x \partial t}) = (\frac{\partial p}{\partial t}) = (\frac{\partial q}{\partial x}); (\frac{\partial \partial z}{\partial t \partial t}) = (\frac{\partial q}{\partial t}).$$

Propofita ergo relatione inter has formulas et praecedentes, fimulque ipfas quantitas x, t et z, aequatio inter ternas iftas quantitates folas x, t et z erui debet. Huiusmodi quaeftiones frequenter occurrunt in Mechanica et Hydraulica, quando motus corporum flexibilium et fluidorum indagatur; ex quo maxime eft optandum, vt haec altera fectio fecundi libri calculi integralis omni cura excolatur. Neque vero opus erit, vt hanc inueftigationem ad differentialia altiora extendamus, cum nul-

lae adhuc quaeſtiones ſint tractatae, quae tanta calculi incrementa deſiderent.

Definitio 5.

24. Si functiones, quae in calculo integrali ex relatione differentialium quaeruntur, algebraice exhiberi nequeant, tum eae vocantur *tranſcendentes*, quandoquidem earum ratio vires Analyſeos communis tranſcendit.

Corollarium. 1.

25. Quoties ergo integratio non ſuccedit, toties functio quae per integrationem quaeritur, pro tranſcendente eſt habenda. Ita ſi formula differentialis $X\partial x$ integrationem non admittit, eius integrale, quod ita indicari ſolet $\int X\partial x$, eſt functio tranſcendens ipſius x.

Corollarium 2.

26. Hinc intelligitur, ſi y fuerit functio tranſcendens ipſius x, viciſſim fore x functionem tranſcendentem ipſius y, atque ex hac conuerſione nouae functiones tranſcendentes oriuntur.

Corollarium 3.

27. Pro variis partibus et ſectionibus calculi integralis naſcuntur etiam plura genera functionum tranſcendentium, quorum adeo numerus in infinitum exſurgit: vnde patet quanta copia omnium quantitatum poſſibilium nobis adhuc ſit ignota.

Scholion 1.

28. Iam ante quam in Analaſin infinitorum penetrauimus, ſpecies quasdam functionum tranſcendentium cognoſcere licuit. Primam ſuppeditauit doctrina logarithmorum: ſi enim y denotet logarithmum ipſius x, vt ſit $y = l\,x$, erit y vtique functio tranſcendens ipſius x, ſicque logarithmi quaſi primam ſpeciem functionum tranſcendentium conſtituunt. Deinde cum ex

aequa-

aequatione $y = lx$ viciſſim ſit $x = e^y$, erit x vtique etiam functio tranſcendens ipſius y: ac tales functiones vocantur exponentiales. Porro autem conſideratio angulorum aliud genus aperuit: veluti ſi angulus, cuius ſinus eſt $= s$, ponatur $= \Phi$, vt ſit $\Phi = \text{Arc. ſin.} s$, nullum eſt dubium, quin Φ ſit functio tranſcendens ipſius s, et quidem infinitiformis: hincque cum conuertendo prodeat $s = \text{ſin.} \Phi$, erit etiam ſinus s functio tranſcendens anguli Φ. Quanquam autem hae functiones tranſcendentes ſine ſubſidio calculi integralis ſunt agnitae, tamen in ipſo quaſi limine calculi integralis ad eas deducimur: earumque indoles ita nobis iam eſt perſpecta, vt propemodum functionibus algebraicis accenſeri queant. Quare etiam perpetuo in calculo integrali, quoties functiones tranſcendentes ibi repertas ad logarithmos vel angulos reuocare licet, eas tanquam algebraicas ſpectare ſolemus.

Scholion 2.

29. Cum calculus integralis ex inuerſione calculi differentialis oriatur, perinde ac reliquae methodi inuerſae ad notitiam noui generis quantitatum nos perducit. Ita ſi a tyrone primorum elementorum nihil praeter notitiam numerorum integrorum poſitiuorum poſtulemus, apprehenſa additione, ſtatim atque ad operationem inuerſam, ſubtractionem ſcilicet, ducitur, notionem numerorum negatiuorum aſſequetur. Deinde multiplicatione tradita, cum ad diuiſionem progreditur, ibi notionem fractionum accipiet. Porro poſtquam euectionem ad poteſtates didicerit, ſi per operationem inuerſam extractionem radicum ſuſcipiat, quoties negotium non ſuccedit, ideam numerorum irrationalium adipiſcetur, haecque cognitio per totam Analyſin communem ſufficiens cenſetur. Simili ergo modo calculus integralis, quatenus integratio non ſuccedit, nouum nobis genus quantitatum tranſcendentium aperit. Non enim, vti

omnium differentialia exhiberi possunt, ita vicissim omnium differentialium integralia exhibere licet.

Scholion 3.

30. Neque vero statim ac primi conatus in integratione expedienda fuerint initi, functiones quaesitae pro transcendentibus sunt habendae; fieri enim saepe solet, vt integrale etiam algebraicum nonnisi per operationes artificiosas obtineri queat. Deinde quando functio quaesita fuerit transcendens, sollicite videndum est, num forte ad species illas simplicissimas logarithmorum vel angulorum reuocari possit, quo casu solutio algebraicae esset aequiparanda. Quod si minus successerit, formam tamen simplicissimam functionum transcendentium, ad quam quaesitam reducere liceat, indagari conueniet. Ad vsum autem longe commodissimum est, vt valores functionum transcendentium vero proxime exhibentur, quem in finem insignis pars calculi integralis in inuestigationem serierum infinitarum impenditur, quae valores earum functionum contineant.

Theorema.

31. Omnes functiones per calculum integralem inventae sunt indeterminatae, ac requirunt determinationem ex natura quaestionis, cuius solutionem suppeditant, petendam.

Demonstratio.

31. Cum semper infinitae dentur functiones, quarum idem est differentiale, siquidem functionis $P + C$, quicunque valor constanti C tribuatur, differentiale idem est $= \partial P$: vicissim etiam proposito differentiali ∂P, integrale est $P + C$, vbi pro C quantitatem constantem quamcunque ponere licet: vnde patet eam functionem, cuius differentiale datur $= \partial P$, esse indeterminatam, cum quantitatem constantem arbitrariam in se inuoluat. Idem etiam eueniat necesse est, si functio ex qua-

quacunque differentialium relatione ſit determinanda, ſemperque complectetur quantitatem conſtantem arbitrariam, cuius nullum veſtigium in relatione differentialium apparuit. Determinabitur ergo huiusmodi functio per calculum integralem inuenta, dum conſtanti illi arbitrariae certus valor tribuitur, quem ſemper natura quaeſtionis, cuius ſolutio ad illam functionem perduxerat, ſuppeditabit.

Corollarium 1.

32. Si ergo functio y ipſius x ex relatione quapiam differentialium definitur, per conſtantem arbitrariam ingreſſam ita determinari poteſt, vt poſito $x = a$ fiat $y = b$: quo facto functio erit determinata, et pro quouis valore ipſi x tributo functio y determinatum obtinebit valorem.

Corollarium 2.

33. Si ex relatione differentialium ſecundi gradus functio y definiatur, binas inuoluet conſtantes arbitrarias, ideoque duplicem determinationem admittit, qua effici poteſt, vt poſito $x = a$, non ſolum y obtineat datum valorem b, ſed etiam ratio $\frac{\partial y}{\partial x}$ dato valori c fiat aequalis.

Corollarium 3.

34. Si y ſit functio binarum variabilium x et t ex relatione differentialium eruta, etiam conſtantem arbitrariam inuoluet, cuius determinatione effici poterit, vt poſito $t = a$, aequatio inter y et x prodeat data, ſeu naturam datae cuiuspiam curuae exprimat.

Scholion.

35. Iſta functionum integralium, ſeu quae per calculum integralem ſunt inuentae, determinatio quouis caſu ex natura quaeſtionis tractatae facile deducitur; neque vlla difficultate laborat, niſi forte praeter neceſſitatem ſolutio ad differentialia

fuerit perducta, cum per Analysin communem erui potuisset: quo casu perinde atque in Algebra quasi radices inutiles ingeruntur. Cum autem haec determinatio tantum in applicatione ad certos casus instituatur, hic vbi integrandi methodum in genere tradimus, integralia in omni amplitudine eruere conabimur; ita vt constantes per integrationem ingressae maneant arbitrariae, neque nisi conditio quaedam vrgeat, eas determinabimus. Caeterum determinatio functionum ipsius x simplicissima est, qua eae casu $x = 0$, ipsae euanescentes redduntur.

Definitio 6.

36. Integrale *completum* exhiberi dicitur, quando functio quaesita omni extensione cum constante arbitraria repraesentatur. Quando autem ista constans iam certo modo est determinata, integrale vocari solet *particulare*.

Corollarium 1.

37. Quouis ergo casu datur vnicum integrale completum; integralia autem particularia infinita exhiberi possunt. Sic differentialis $x\,\partial x$ integrale completum est $\frac{1}{2}xx + C$, integralia autem particularia $\frac{1}{2}xx$; $\frac{1}{2}xx + 1$; $\frac{1}{2}xx + 2$ etc. multitudine infinita.

Corollarium 2.

38. Integrale ergo completum omnia integralia particularia in se complectitur; ex eoque haec omnia facile formari possunt. Vicissim autem ex integralibus particularibus, integrale completum non innotescit. Saepenumero autem, vti deinceps patebit, habetur methodus, ex integrali particulari completum inueniendi.

Scholion.

39. Interdum facile est integrale particulare coniectura vel diuinatione assequi. Veluti si eiusmodi functio ipsius x, quae

quae ſit y, quaeritur, vt ſit $\partial y + yy\,\partial x = \partial x + xx\,\partial y$, huic aequationi manifeſto ſatisfit ſumendo $y = x$; quod ergo eſt integrale particulare, quoniam, in eo nulla ineſt conſtans arbitraria: at integrale completum reperitur $y = \frac{1 + Cx}{C + x}$, quod illud particulare in ſe continet, ſumendo $C = \infty$. Simili modo ſumendo $C = 0$, hinc aliud integrale obtinetur $y = \frac{1}{x}$, quod ſuperiori aequationi perinde ſatisfacit ac prius $y = x$. Omnia autem integralia particularia, quaecunque ſatisfaciunt, contineri neceſſe eſt in formula generali $y = \frac{1 + Cx}{C + x}$, prouti conſtanti arbitrariae C alii atque alii valores tribuantur: ita ſumto $C = 1$ ſit etiam $y = 1$. Plerumque autem euenire ſolet, vt etiamſi integrale quoddam particulare ſit algebraicum, tamen integrale completum ſit tranſcendens. Veluti ſi propoſita ſit haec aequatio $\partial y + y\,\partial x = \partial x + x\,\partial x$, ſtatim patet ſatisfieri poſito $y = x$, quod ergo eſt integrale particulare; verum integrale completum conſtantem arbitrariam C inuoluens, eſt $y = x + Ce^{-x}$, denotante e numerum cuius logarithmus $= 1$: niſi ergo hic ſumatur $C = 0$, functio y ſemper eſt tranſcendens. Haec in genere notaſſe ſufficiat, antequam ad tractationem ipſam calculi integralis aggrediamur, quandoquidem ad omnes integrationes pertinent: nunc igitur forma tractationis expoſita, ad opus tractandum pergamus.

CON-

CONSPECTVS
VNIVERSI OPERIS
DE
CALCVLO INTEGRALI.

LIBER PRIOR: Tradit methodum inueſtigandi functiones vnius variabilis ex data quadam relatione differentialium, continetque duas partes:

Pars prior: Quando relatio illa data tantum differentialia primi gradus complectitur.

Pars poſterior: Quando relatio illa data differentialia ſecundi altiorumue graduum complectitur.

LIBER POSTERIOR: Tradit methodum inueſtigandi functiones duarum pluriumue variabilium ex data quadam relatione differentialium, continetque duas partes:

Pars prior: Quando relatio illa data tantum differentialia primi gradus complectitur.

Pars poſterior: Quando relatio illa data differentialia ſecundi altiorumue graduum complectitur.

CAL-

CALCVLI INTEGRALIS
LIBER PRIOR.

PARS PRIMA,

SEV

METHODVS INVESTIGANDI FVNCTIONES VNIVS VARIABILIS EX DATA RELATIONE QVACVNQVE DIFFERENTIALIVM PRIMI GRADVS.

SECTIO PRIMA,

DE

INTEGRATIONE FORMVLARVM DIFFERENTIALIVM.

CAPVT I.

DE INTEGRATIONE FORMVLARVM DIFFERENTIALIVM RATIONALIVM.

Definitio.

40.

Formula differentialis *rationalis* eſt, quando variabilis x, cuius functio quaeritur, differentiale ∂x multiplicatur in functionem rationalem ipſius x: ſeu ſi X deſignet functionem rationalem ipſius x, haec formula differentialis $X \partial x$ dicitur rationalis.

Corollarium 1.

41. In hoc ergo capite eiusmodi functio ipſius x quaeritur, quae ſi ponatur y, vt $\frac{\partial y}{\partial x}$ aequetur functioni rationali ipſius x, ſeu poſita tali functione $= X$, vt ſit $\frac{\partial y}{\partial x} = X$.

Corollarium 2.

42. Hinc quaeritur eiusmodi functio ipſius x, cuius differentiale ſit $= X \partial x$; huius ergo integrale, quod ita indicari ſolet $\int X \partial x$, praebebit functionem quaeſitam.

Corollarium 3.

43. Quodſi P fuerit eiusmodi functio ipſius x, vt eius differentiale ∂P ſit $= X \partial x$, quoniam quantitatis $P + C$ idem eſt differentiale, formulae propoſitae $X \partial x$ integrale completum eſt $P + C$.

Scholion 1.

44. Ad libri primi partem priorem huiusmodi referuntur quaestiones, quibus functiones solius variabilis x, ex data differentialium primi gradus relatione quaeruntur. Scilicet si functio quaesita $= y$ et $\frac{\partial y}{\partial x} = p$, id praestari oportet, vt proposita aequatione quacunque inter ternas quantitates x, y et p, inde indoles functionis y, seu aequatio inter x et y, elisa littera p, inueniatur. Quaestio autem sic in genere proposita vires analyseos adeo superare videtur, vt eius solutio nunquam expectari queat. In casibus igitur simplicioribus vires nostrae sunt exercendae, inter quos primum occurrit casus, quo p functioni cuipiam ipsius x puta X aequatur, vt sit $\frac{\partial y}{\partial x} = X$, seu $\partial y = X \partial x$, ideoque integrale $y = \int X \partial x$ requiratur, in quo primam sectionem collocamus. Verum et hic casus pro varia indole functionis X latissime patet, ac plurimis difficultatibus implicatur: vnde in hoc capite eiusmodi tantum quaestiones euoluere instituimus, in quibus ista functio X est rationalis: deinceps ad functiones irrationales atque adeo transcendentes progressuri. Hinc ista pars commode in duas sectiones subdiuiditur, in quarum altera integratio formularum simplicium, quibus $p = \frac{\partial y}{\partial x}$ functioni tantum ipsius x aequatur, est tradenda, in altera autem rationem integrandi doceri conueniet, cum proposita fuerit aequatio quaecunque ipsarum x, y et p. Et cum in his duabus sectionibus, ac potissimum priore, a Geometris plurimum sit elaboratum, eae maximam partem totius operis complebunt.

Scholion 2.

45. Prima autem integrationis principia ex ipso calculo differentiali sunt petenda, perinde ac principia diuisionis ex multiplicatione, et principia extractionis radicum ex ratione euectio-

cuectionis ad poteſtates ſumi ſolent. Cum igitur ſi quantitas differentianda ex pluribus partibus conſtet, vt $P+Q-R$, eius differentiale ſit $\partial P+\partial Q-\partial R$, ita viciſſim ſi formula differentialis ex pluribus partibus conſtet, vt $P\partial x+Q\partial x-R\partial x$, integrale erit $\int P\partial x+\int Q\partial x-\int R\partial x$, ſingulis ſcilicet partibus ſeorſim integrandis. Deinde cum quantitatis aP differentiale ſit $a\partial P$, formulae differentialis $aP\partial x$ integrale erit $a\int P\partial x$: ſcilicet per quam quantitatem conſtantem formula differentialis multiplicatur, per eandem integrale multiplicari debet. Ita ſi formula differentialis ſit $aP\partial x+bQ\partial x+cR\partial x$, quaecunque functiones ipſius x litteris P, Q, R deſignentur, integrale erit $a\int P\partial x+b\int Q\partial x+c\int R\partial x$: ita vt integratio tantum in ſingulis formulis $P\partial x$, $Q\partial x$ et $R\partial x$, ſit inſtituenda. Hocque facto inſuper adiici debet conſtans arbitraria C, vt integrale completum obtineantur.

Problema 1.

46. Inuenire functionem ipſius x, vt eius differentiale ſit $=ax^n\partial x$, ſeu integrare formulam differentialem $ax^n\partial x$.

Solutio.

Cum poteſtatis x^m differentiale ſit $mx^{m-1}\partial x$, erit viciſſim:

$\int mx^{m-1}\partial x=m\int x^{m-1}\partial x=x^m$, ideoque $\int x^{m-1}\partial x=\frac{1}{m}x^m$.

Fiat $m-1=n$, ſeu $m=n+1$, erit:

$$\int x^n\partial x=\frac{1}{n+1}x^{n+1}, \text{ et } a\int x^n\partial x=\frac{a}{n+1}x^{n+1}.$$

Vnde formulae differentialis propoſitae $ax^n\partial x$ integrale completum erit $\frac{a}{n+1}x^{n+1}+C$, cuius ratio vel inde patet, quod eius differentiale reuera ſit $=ax^n\partial x$. Atque haec integratio ſemper locum habet, quicunque numerus exponenti n tribuatur, ſiue poſitiuus ſiue negatiuus, ſiue integer ſiue fractus, ſiue etiam irrationalis.

Vnicus casus hinc excipitur, quo est exponens $n = -1$, seu haec formula $\frac{a\partial x}{x}$ integranda proponitur. Verum in calculo differentiali iam ostendimus, si $l\,x$ denotet logarithmum hyperbolicum ipsius x, fore eius differentiale $= \frac{\partial x}{x}$; vnde vicissim concludimus esse $\int \frac{\partial x}{x} = l\,x$, et $\int \frac{a\partial x}{x} = a\,l\,x$. Quare adiecta constante arbitraria, erit formulae $\frac{a\partial x}{x}$ integrale completum $= a\,l\,x + C = l\,x^a + C$: quod etiam pro C ponendo $l\,c$, ita exprimitur $l\,c\,x^a$.

Corollarium 1.

47. Formulae ergo differentialis $ax^n\partial x$ integrale semper est algebraicum, solo excepto casu quo $n = -1$, et integrale per logarithmos exprimitur, qui ad functiones transcendentes sunt referendi. Est scilicet $\int \frac{a\partial x}{x} = alx + C = lcx^a$.

Corollarium 2.

48. Si exponens n numeros positiuos denotet, sequentes integrationes vtpote maxime obuiae probe sunt tenendae:

$$\int a\partial x = ax + C;\ \int ax\partial x = \tfrac{a}{2}xx + C;\ \int ax^2\partial x = \tfrac{a}{3}x^3 + C;$$
$$\int ax^3\partial x = \tfrac{a}{4}x^4 + C;\ \int ax^4\partial x = \tfrac{a}{5}x^5 + C;\ \int ax^5\partial x = \tfrac{a}{6}x^6 + C.$$

Corollarium 3.

49. Si n sit numerus negatiuus, posito $n = -m$, fit

$$\int \frac{a\,\partial x}{x^m} = \frac{a}{1-m}\,x^{1-m} + C = \frac{-a}{(m-1)\,x^{m-1}} + C;$$

vnde hi casus simpliciores notentur:

$$\int \frac{a\partial x}{x^2} = \frac{-a}{x} + C;\ \int \frac{a\partial x}{x^3} = \frac{-a}{2xx} + C;\ \int \frac{a\partial x}{x^4} = \frac{-a}{3x^3} + C;$$
$$\int \frac{a\partial x}{x^5} = \frac{-a}{4x^4} + C;\ \int \frac{a\partial x}{x^6} = \frac{-a}{5x^5} + C;\ \text{etc.}$$

Corol-

Corollarium 4.

50. Quin etiam si n denotet numeros fractos, integralia hinc obtinentur. Sit primo $n = \frac{m}{2}$, erit

$$\int a\,\partial x \sqrt{x^m} = \frac{2a}{m+2}\, x \sqrt{x^m} + C.$$

Vnde casus notentur:

$$\int a\,\partial x \sqrt{x} = \tfrac{2a}{3} x \sqrt{x} + C;\ \int a x\,\partial x \sqrt{x} = \tfrac{2a}{5} x^2 \sqrt{x} + C;$$
$$\int a x x\,\partial x \sqrt{x} = \tfrac{2a}{7} x^3 \sqrt{x} + C;\ \int a x^3\,\partial x \sqrt{x} = \tfrac{2a}{9} x^4 \sqrt{x} + C.$$

Corollarium 5.

51. Ponatur etiam $n = \frac{-m}{2}$, et habebitur

$$\int \frac{a\,\partial x}{\sqrt{x^m}} = \frac{2a}{2-m} \cdot \frac{x}{\sqrt{x^m}} + C = \frac{-2a}{(m-2)\sqrt{x^{m-2}}} + C.$$

Vnde hi casus notentur:

$$\int \tfrac{a\,\partial x}{\sqrt{x}} = 2a\sqrt{x} + C;\ \int \tfrac{a\,\partial x}{x\sqrt{x}} = \tfrac{-2a}{\sqrt{x}} + C;$$
$$\int \tfrac{a\,\partial x}{xx\sqrt{x}} = \tfrac{-2a}{3x\sqrt{x}} + C;\ \int \tfrac{a\,\partial x}{x^3\sqrt{x}} = \tfrac{-2a}{5x^2\sqrt{x}} + C.$$

Corollarium 6.

52. Si in genere ponamus $n = \frac{\mu}{\nu}$, fiet:

$$\int a x^{\frac{\mu}{\nu}} \partial x = \tfrac{\nu a}{\mu+\nu} x^{\frac{\mu+\nu}{\nu}} + C,$$ seu per radicalia

$$\int a\,\partial x \sqrt[\nu]{x^\mu} = \tfrac{\nu a}{\mu+\nu} \sqrt[\nu]{x^{\mu+\nu}} + C.$$

Sin autem ponatur $n = \frac{-\mu}{\nu}$ habebitur:

$$\int \frac{a\,\partial x}{x^{\frac{\mu}{\nu}}} = \frac{\nu a}{\nu-\mu} x^{\frac{\nu-\mu}{\nu}} + C,$$ seu per radicalia:

$$\int \frac{a\,\partial x}{\sqrt[\nu]{x^\mu}} = \frac{\nu a}{\nu-\mu} \sqrt[\nu]{x^{\nu-\mu}} + C.$$

Scho-

Scholion 1.

53. Quanquam in hoc capite functiones tantum rationales tractare institueram, tamen istae irrationalitates tam sponte se obtulerunt, vt perinde ac rationales tractari possint. Caeterum hinc quoque formulae magis complicatae integrari possunt, si pro x functiones alius cuiuspiam variabilis z statuantur. Veluti si ponamus $x = f + gz$, erit $\partial x = g\partial z$: quare si pro a scribamus $\frac{a}{g}$, habebitur:

$$\int a\partial z(f+gz)^n = \frac{a}{(n+1)g}(f+gz)^{n+1} + C.$$

Casu autem singulari, quo $n = -1$:

$$\int \frac{a\partial z}{f+gz} = \frac{a}{g}l(f+gz) + C.$$

Tum si sit $n = -m$, fiet:

$$\int \frac{a\partial z}{(f+gz)^m} = \frac{-a}{(m-1)g(f+gz)^{m-1}} + C.$$

Ac posito $n = \frac{\mu}{\nu}$, prodit:

$$\int a\partial z(f+gz)^{\frac{\mu}{\nu}} = \frac{\nu a}{(\nu+\mu)g}(f+gz)^{\frac{\mu}{\nu}+1} + C.$$

Posito autem $n = -\frac{\mu}{\nu}$, obtinetur,

$$\int \frac{a\partial z}{(f+gz)^{\frac{\mu}{\nu}}} = \frac{\nu a(f+gz)}{(\nu-\mu)g(f+gz)^{\frac{\mu}{\nu}}} + C.$$

Scholion 2.

54. Caeterum hic insignis proprietas annotari meretur. Cum hic quaeratur functio y, vt sit $\partial y = ax^n\partial x$, si ponamus $\frac{\partial y}{\partial x} = p$, haec habebitur relatio $p = ax^n$, ex qua functio y inuestigari debet. Quoniam igitur est

$$y = \frac{a}{n+1}x^{n+1} + C,$$

ob

ob $ax^n = p$, erit quoque $y = \frac{px}{n+1} + C$: ficque cafum habemus, vbi relatio differentialium per aequationem quandam inter x, y et p proponitur, cuique iam nouimus fatisfieri per aequationem $y = \frac{a}{n+1} x^{n+1} + C$. Verum haec non amplius erit integrale completum pro relatione in aequatione $y = \frac{px}{n+1} + C$ contenta, fed tantum particulare, quoniam integrale illud non inuoluit nouam conftantem, quae in relatione differentiali non infit. Integrale autem completum eft $y = \frac{aD}{n+1} x^{n+1} + C$: nouam conftantem D inuoluens: hinc enim fit $\frac{\partial y}{\partial x} = aDx^n = p$, ideoque $y = \frac{px}{n+1} + C$. Etfi hoc non ad praefens inftitutum pertinet, tamen notaffe iuuabit.

Problema 2.

55. Inuenire functionem ipfius x, cuius differentiale fit $= X \partial x$, denotante X functionem quamcunque rationalem integram ipfius x, feu definire integrale $\int X \partial x$.

Solutio.

Cum X fit functio rationalis integra ipfius x, in hac forma contineatur neceffe eft:

$$X = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 \text{ etc.}$$

vnde per problema precedens integrale quaefitum eft

$$\int X \partial x = C + \alpha x + \tfrac{1}{2}\beta x^2 + \tfrac{1}{3}\gamma x^3 + \tfrac{1}{4}\delta x^4 + \tfrac{1}{5}\varepsilon x^5 + \tfrac{1}{6}\zeta x^6 + \text{ etc.}$$

Atque in genere fi fit $X = \alpha x^\lambda + \beta x^\mu + \gamma x^\nu + \text{etc.}$ erit

$$\int X \partial x = C + \frac{\alpha}{\lambda+1} x^{\lambda+1} + \frac{\beta}{\mu+1} x^{\mu+1} + \frac{\gamma}{\nu+1} x^{\nu+1} \text{ etc.}$$

vbi exponentes λ, μ, ν etc. etiam numeros tam negatiuos quam fractos fignificare poffunt; dummodo notetur, fi fuerit $\lambda = -1$, fore $\int \frac{\alpha \partial x}{x} = \alpha l x$, qui eft vnicus cafus ad ordinem tranfcendentium referendus.

D Pro-

Problema 3.

56. Si X denotet functionem quamcunque rationalem fractam ipsius x, methodum describere, cuius ope formulae $X \partial x$ integrale inuestigari conueniat.

Solutio.

Sit igitur $X = \frac{M}{N}$, ita vt M et N futurae sint functiones integrae ipsius x, ac primo dispiciatur, num summa potestas ipsius x in numeratore M tanta sit, vel etiam maior quam in denominatore N? quo casu ex fractione $\frac{M}{N}$ partes integrae per diuisionem eliciantur, quarum integratio, cum nihil habeat difficultatis, totum negotium reducitur ad eiusmodi fractionem $\frac{M}{N}$, in cuius numeratore M summa potestas ipsius x minor sit quam denominatore N.

Tum quaerantur omnes factores ipsius denominatoris N, tam simplices si fuerint reales, quam duplices reales, vicem scilicet binorum simplicium imaginariorum gerentes; simulque videndum est, vtrum hi factores omnes sint inaequales nec ne? pro factorum enim aequalitate alio modo resolutio fractionis $\frac{M}{N}$ in fractiones simplices est instituenda, quandoquidem ex singulis factoribus fractiones partiales nascuntur, quarum aggregatum fractioni propositae $\frac{M}{N}$ aequatur. Scilicet ex factore simplici $a + bx$ nascitur fractio $\frac{A}{a+bx}$; si bini sint aequales, seu denominator N factorem habeat $(a+bx)^2$, hinc nascuntur fractiones $\frac{A}{(a+bx)^2} + \frac{B}{a+bx}$; ex huiusmodi autem factore $(a+bx)^3$ hae tres fractiones

$$\frac{A}{(a+bx)^3} + \frac{B}{(a+bx)^2} + \frac{C}{a+bx}$$

et ita porro.

Factor autem duplex, cuius forma est $aa - 2abx\,\text{cos.}\,\zeta + bbxx$, nisi alius ipsi fuerit aequalis, dabit fractionem partialem

tialem $\frac{A + Bx}{aa - 2abx\,\mathrm{cos}.\zeta + bbxx}$: si autem denominator N duos huiusmodi factores aequales inuoluat, inde nascuntur binae huiusmodi fractiones partiales:

$$\frac{A + Bx}{(aa - 2abx\,\mathrm{cos}.\zeta + bbxx)^2} + \frac{C + Dx}{aa - 2abx\,\mathrm{cos}.\zeta + bbxx}:$$

at si cubus adeo $(aa - 2abx\,\mathrm{cos}.\zeta + bbxx)^3$ fuerit factor denominatoris N, ex eo oriuntur huiusmodi tres fractiones partiales:

$$\frac{A + Bx}{(aa - 2abx\,\mathrm{cos}.\zeta + bbxx)^3} + \frac{C + Dx}{(aa - 2abx\,\mathrm{cos}.\zeta + bbxx)^2}$$
$$+ \frac{E + Fx}{aa - 2abx\,\mathrm{cos}.\zeta + bbxx}$$

et ita porro.

Cum igitur hoc modo fractio proposita $\frac{M}{N}$ in omnes suas fractiones simplices fuerit resoluta, omnes continebuntur in alterutra harum formarum,

$$\text{vel } \frac{A}{(a + bx)^n}, \text{ vel } \frac{A + Bx}{(aa - 2abx\,\mathrm{cos}.\zeta + bbxx)^n},$$

ac singulos iam per ∂x multiplicatos integrari oportet, erit omnium horum integralium aggregatum valor functionis quaesitae $\int X\partial x = \int \frac{M}{N}\partial x$.

Corollarium 1.

57. Pro integratione ergo omnium huiusmodi formularum $\frac{M}{N}\partial x$, totum negotium reducitur ad integrationem huiusmodi binarum formularum:

$$\int \frac{A\partial x}{(a + bx)^n} \text{ et } \int \frac{(A + Bx)\partial x}{(aa - 2abx\,\mathrm{cos}.\zeta + bbxx)^n},$$

dum pro n successiue scribuntur numeri 1, 2, 3, 4 etc.

Corollarium 2.

58. Ac prioris quidem formae integrale iam supra (53) est expeditum, vnde patet fore:

$$\int \frac{A\,\partial x}{a+bx} = \frac{A}{b}\, l\,(a+bx) + \text{Conft.}$$
$$\int \frac{A\,\partial x}{(a+bx)^2} = \frac{-A}{b\,(a+bx)} + \text{Conft.}$$
$$\int \frac{A\,\partial x}{(a+bx)^3} = \frac{-A}{2b\,(a+bx)^2} + \text{Conft.}$$

et generatim :

$$\int \frac{A\,\partial x}{(a+bx)^n} = \frac{-A}{(n-1)\,b\,(a+bx)^{n-1}} + \text{Conft.}$$

Corollarium 3.

59. Ad propofitum ergo abfoluendum nihil aliud fupereft, nifi vt integratio huius formulae

$$\int \frac{(A+Bx)\,\partial x}{(aa-2abx\,\text{cof.}\,\zeta+bbxx)^n}$$

doceatur, primo quidem cafu $n=1$, tum vero cafibus $n=2$, $n=3$, $n=4$, etc.

Scholion 1.

60. Nifi vellemus imaginaria euitare, totum negotium ex iam traditis confici poffet: denominatore enim N in omnes fuos factores fimplices refoluto, fiue fint reales fiue imaginarii, fractio propofita femper refolui poterit in fractiones partiales huius formae $\frac{A}{a+bx}$, vel huius $\frac{A}{(a+bx)^n}$, quarum integralia cum fint in promptu, totius formae $\frac{M}{N}\partial x$, integrale habetur. Tum autem non parum moleftum foret binas partes imaginarias ita coniungere, vt expreffio realis refultaret, quod tamen rei natura abfolute exigit.

Scholion 2.

61. Hic vtique poftulamus, refolutionem cuiusque functionis integrae in factores nobis concedi, etiamfi algebra neutiquam adhuc eo fit perducta, vt haec refolutio actu inftitui poffit.

possit. Hoc autem in Analysi vbique postulari solet, vt quo longius progrediamur, ea quae retro sunt relicta, etiamsi non satis fuerint explorata, tanquam congnita assumamus: sufficere scilicet hic potest, omnes factores per methodum approximationum quantumuis prope assignari posse. Simili modo cum in calculo integrali longius processerimus, integralia omnium huiusmodi formularum $X \partial x$, quaecunque functio ipsius x littera X significetur, tanquam cognita spectabimus; plurimumque nobis praestitisse videbimur, si integralia magis abscondita ad eas formas reducere valuerimus: atque hoc etiam in vsu practico nihil turbat, cum valores talium formularum $\int X \partial x$, quantumuis prope assignare liceat, vti in sequentibus ostendemus. Caeterum ad has integrationes, resolutio denominatoris N in suos factores absolute est necessaria, propterea quod singuli hi factores in expressionem integralis ingrediuntur: paucissimi sunt casus, iique maxime obuii, quibus ista resolutione carere possumus: veluti si proponatur haec formula $\frac{x^{n-1}\partial x}{1+x^n}$, statim patet, posito $x^n = v$, eam abire in $\frac{\partial v}{n(1+v)}$, cuius integrale est $\frac{1}{n} l(1+v) = \frac{1}{n} l(1+x^n)$; vbi resolutione in factores non fuerat opus. Verum huiusmodi casus per se tam sunt perspicui, vt eorum tractatio nulla peculiari explicatione indigeat.

Problema 4.

62. Inuenire integrale huius formulae:

$$y = \int \frac{(A + Bx)\partial x}{aa - 2abx\,cos.\zeta + bbxx}.$$

Solutio.

Cum numerator duabus constet partibus $A\partial x + Bx\partial x$, haec posterior $Bx\partial x$ sequenti modo tolli poterit. Cum sit

$$l(aa-2abx\,\mathrm{cof.}\zeta+bbxx)=\int\frac{-2ab\,\partial x\,\mathrm{cof.}\zeta+2bbx\,\partial x}{aa-2abx\,\mathrm{cof.}\zeta+bbxx},$$

multiplicetur haec aequatio per $\frac{B}{2bb}$, et a propofita auferatur: fic enim prodibit

$$y-\frac{B}{2bb}l(aa-2abx\,\mathrm{cof.}\zeta+bbxx)=\int\frac{(A+\frac{Ba\,\mathrm{cof.}\zeta}{b})\partial x}{aa-2abx\,\mathrm{cof.}\zeta+bbxx}:$$

ita vt haec tantum formula integranda fuperfit. Ponatur brevitatis gratia $A+\frac{Ba\,\mathrm{cof.}\zeta}{b}=C$, vt habeatur haec formula:

$$\int\frac{C\,\partial x}{aa-2abx\,\mathrm{cof.}\zeta+bbxx},$$

quae ita exhiberi poteft

$$\int\frac{C\,\partial x}{aa\,\mathrm{fin.}\zeta^2+(bx-a\,\mathrm{cof.}\zeta)^2}.$$

Statuatur $bx-a\,\mathrm{cof.}\zeta=av\,\mathrm{fin.}\zeta$, hincque $\partial x=\frac{a\,\partial v\,\mathrm{fin.}\zeta}{b}$: vnde formula noftra erit:

$$\int\frac{Ca\,\partial v\,\mathrm{fin.}\zeta:b}{aa\,\mathrm{fin.}\zeta^2(1+vv)}=\frac{C}{ab\,\mathrm{fin.}\zeta}\int\frac{\partial v}{1+vv}.$$

Ex calculo autem differentiali nouimus effe:

$$\int\frac{\partial v}{1+vv}=\mathrm{Arc.\,tang.}\,v=\mathrm{Arc.\,tang.}\frac{bx-a\,\mathrm{cof.}\zeta}{a\,\mathrm{fin.}\zeta};$$

vnde ob $C=\frac{Ab+Ba\,\mathrm{cof.}\zeta}{b}$, erit noftrum integrale

$$\frac{Ab+Ba\,\mathrm{cof.}\zeta}{abb\,\mathrm{fin.}\zeta}\,\mathrm{Arc.\,tang.}\frac{bx-a\,\mathrm{cof.}\zeta}{a\,\mathrm{fin.}\zeta}.$$

Quocirca formulae propofitae $\frac{(A+Bx)\partial x}{aa-2abx\,\mathrm{cof.}\zeta+bbxx}$ integrale eft:

$$\frac{B}{2bb}l(aa-2abx\,\mathrm{cof.}\zeta+bbxx)+\frac{Ab+Ba\,\mathrm{cof.}\zeta}{abb\,\mathrm{fin.}\zeta}\,\mathrm{Arc.\,tang.}\frac{bx-a\,\mathrm{cof.}\zeta}{a\,\mathrm{fin.}\zeta},$$

quod vt fiat completum, conftans arbitraria C infuper addatur.

Corollarium I

63. Si ad Arc. tang. $\frac{bx-a\,\mathrm{cof.}\zeta}{a\,\mathrm{fin.}\zeta}$ addamus Arc. tang. $\frac{\mathrm{cof.}\zeta}{\mathrm{fin.}\zeta}$, quippe qui in conftante addenda contentus concipiatur, prodibit Arc. tang. $\frac{bx\,\mathrm{fin.}\zeta}{a-bx\,\mathrm{cof.}\zeta}$, ficque habebimus:

$\int$

$$\int\frac{(A+Bx)\partial x}{aa-2abx\,cof.\zeta+bbxx}=\frac{B}{2bb}l(aa-2abx\,cof.\zeta+bbxx)$$
$$+\frac{Ab+Ba\,cof.\zeta}{abb\,fin.\zeta}\text{ Arc. tang. }\frac{bx\,fin.\zeta}{a-bx\,cof.\zeta}$$

adiecta constante C.

Corollarium 2.

64. Si velimus vt integrale hoc euanescat, posito $x=0$, constans C sumi debet $=-\frac{B}{2bb}l\,aa$, sicque fiet:

$$\int\frac{(A+Bx)\partial x}{aa-2abx\,cof.\zeta+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa-2abx\,cof.\zeta+bbxx)}}{a}$$
$$+\frac{Ab+Ba\,cof.\zeta}{abb\,fin.\zeta}\text{ Arc. tang. }\frac{bx\,fin.\zeta}{a-bx\,cof.\zeta}.$$

Pendet ergo hoc integrale partim a logarithmis, partim ab arcubus circularibus seu angulis.

Corollarium 3.

65. Si littera B euanescat, pars a logarithmis pendens euanescit, sitque

$$\int\frac{A\partial x}{aa-2abx\,cof.\zeta+bbxx}=\frac{A}{ab\,fin.\zeta}\text{ Arc. tang. }\frac{bx\,fin.\zeta}{a-bx\,cof.\zeta}+C$$

sicque per solum angulum definitur.

Corollarium 4.

66. Si angulus ζ sit rectus, ideoque $cof.\zeta=0$, et $fin.\zeta=1$, habebitur:

$$\int\frac{(A+Bx)\partial x}{aa+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+bbxx)}}{a}+\frac{A}{ab}\text{ Arc. tang. }\frac{bx}{a}+C.$$

Si angulus ζ sit 60°, ideoque $cof.\zeta=\frac{1}{2}$ et $fin.\zeta=\frac{\sqrt{3}}{2}$, erit:

$$\int\frac{(A+Bx)\partial x}{aa-abx+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa-abx+bbxx)}}{a}+\frac{2Ab+Ba}{abb\sqrt{3}}\text{ Arc. tang. }\frac{bx\sqrt{3}}{2a-bx}.$$

At si $\zeta=120°$, ideoque $cof.\zeta=-\frac{1}{2}$ et $fin.\zeta=\frac{\sqrt{3}}{2}$, erit:

$$\int\frac{(A+Bx)\partial x}{aa+abx+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+abx+bbxx)}}{a}+\frac{2Ab-Ba}{abb\sqrt{3}}\text{ Arc. tang. }\frac{bx\sqrt{3}}{2a+bx}.$$

Scho-

Scholion 1.

67. Omnino hic notatu dignum euenit, quod casu $\zeta = 0$, quo denominator $aa - 2abx + bbxx$ fit quadratum, ratio anguli ex integrali discedat. Posito enim angulo ζ infinite paruo, erit cos. $\zeta = 1$ et sin. $\zeta = \zeta$; vnde pars logarithmica fit $\frac{B}{bb} l \frac{a-bx}{a}$, et altera pars:

$$\frac{Ab + Ba}{abb\zeta} \text{ Arc. tang. } \frac{bx\zeta}{a-bx} = \frac{(Ab+Ba)x}{ab(a-bx)},$$

quia arcus infinite parui $\frac{bx\zeta}{a-bx}$ tangens ipsi est aequalis, sicque haec pars fit algebraica. Quocirca erit

$$\int \frac{(A + Bx)\partial x}{(a-bx)^2} = \frac{B}{bb} l \frac{a-bx}{a} + \frac{(Ab+Ba)x}{ab(a-bx)} + \text{Const.}$$

cuius veritas ex praecedentibus est manifesta: est enim

$$\frac{A+Bx}{(a-bx)^2} = -\frac{B}{b(a-bx)} + \frac{Ab+Ba}{b(a-bx)^2}.$$

Iam vero est

$$\int \frac{-B\partial x}{b(a-bx)} = \frac{B}{bb} l(a-bx) - \frac{B}{bb} l a = \frac{B}{bb} l \frac{a-bx}{a},$$

$$\int \frac{(Ab+Ba)\partial x}{b(a-bx)^2} = \frac{Ab+Ba}{bb(a-bx)} - \frac{(Ab+Ba)}{abb} = \frac{(Ab+Ba)x}{ab(a-bx)},$$

siquidem vtraque integratio ita determinetur vt, casu $x = 0$, integralia euanescant.

Scholion 2.

68. Simili modo, quo hic vsi sumus, si in formula differentiali fracta $\frac{M\partial x}{N}$, summa potestas ipsius x, in numeratore M, vno gradu minor sit quam in denominatore N, etiam is terminus tolli poterit. Sit enim

$$M = A x^{n-1} + B x^{n-2} + C x^{n-3} \text{ etc. et}$$
$$N = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} \text{ etc.}$$

ac ponatur $\frac{M\partial x}{N} = \partial y$: Cum iam sit

∂ N

$\partial N = n\alpha x^{n-1}\partial x + (n-1)\beta x^{n-2}\partial x + (n-2)\gamma x^{n-3}\partial x$ etc. erit:

$\frac{A\,\partial N}{n\alpha N} = \frac{\partial x}{N}(A\,x^{n-1} + \frac{(n-1)A\beta}{n\alpha}x^{n-2} + \frac{(n-2)A\gamma}{n\alpha}x^{n-3})$ etc.

quo valore inde fubtracto remanebit:

$\partial y - \frac{A\,\partial N}{n\alpha N} = \frac{\partial x}{N}[(B - \frac{(n-1)A\beta}{n\alpha})x^{n-2} + (C - \frac{(n-2)A\gamma}{n\alpha})x^{n-3}$ etc.$]$

Quare fi breuitatis gratia ponatur:

$B - \frac{(n-1)A\beta}{n\alpha} = \mathfrak{B};\ C - \frac{(n-2)A\gamma}{n\alpha} = \mathfrak{C};\ D - \frac{(n-3)A\delta}{n\alpha} = \mathfrak{D}$ etc.

obtinebitur:

$$y = \frac{A}{n\alpha}lN + \int \frac{\partial x(\mathfrak{B}x^{n-2} + \mathfrak{C}x^{n-3} + \mathfrak{D}x^{n-4} + \text{etc.})}{\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3}\ \text{etc.}} = \int \frac{M\,\partial x}{N}.$$

Hoc igitur modo omnes formulae differentiales fractae eo reduci poffunt, vt fumma poteftas ipfius x, in numeratore duobus pluribusue gradibus minor fit quam in denominatore.

Problema 5.

69. Formulam integralem $\int \frac{(A+Bx)\partial x}{(aa - 2abx\cos.\zeta + bbxx)^{n+1}}$ ad aliam fimilem reducere, vbi poteftas denominatoris fit vno gradu inferior.

Solutio.

Sit breuitatis gratia $aa - 2abx\cos.\zeta + bbxx = X$, ac ponatur $\int \frac{(A+Bx)\partial x}{X^{n+1}} = y$. Cum ob $\partial X = -2ab\partial x\cos.\zeta + 2bbx\partial x$, fit:

$$\partial \frac{C+Dx}{X^n} = -\frac{n(C+Dx)\partial X}{X^{n+1}} + \frac{D\partial x}{X^n}$$

ideoque:

$$\frac{C+Dx}{X^n} = \int \frac{2nb(C+Dx)(a\cos.\zeta - bx)\partial x}{X^{n+1}} + \int \frac{D\partial x}{X^n}:$$

habebimus:

$$y + \frac{C + Dx}{X^n} = \int \frac{\partial x[A + 2nCab\,\mathrm{cos}.\zeta + x(B + 2nDab\,\mathrm{cos}.\zeta - 2nCbb) - 2nDbbxx]}{X^{n+1}}$$

$$+ \int \frac{D\,\partial x}{X^n}.$$

Iam in formula priori litterae C et D ita definiantur, vt numerator per X fiat diuisibilis. Oportet ergo sit $= -2nDX\partial x$, vnde nanciscimur:

$$A + 2nCab\,\mathrm{cos}.\zeta = -2nDaa, \text{ et}$$

$$B + 2nDab\,\mathrm{cos}.\zeta - 2nCbb = 4nDab\,\mathrm{cos}.\zeta,$$

seu $B - 2nCbb = 2nDab\,\mathrm{cos}.\zeta$; hincque

$$2nDa = \frac{B - 2nCbb}{b\,\mathrm{cos}.\zeta}.$$

At ex priori conditione est

$2nDa = \frac{-A - 2nCab\,\mathrm{cos}.\zeta}{a}$, quibus aequatis fit:

$Ba + Ab\,\mathrm{cos}.\zeta - 2nCabb\,\mathrm{sin}.\zeta^2 = 0$, seu

$C = \frac{Ba + Ab\,\mathrm{cos}.\zeta}{2nabb\,\mathrm{sin}.\zeta^2}$: vnde

$$B - 2nCbb = \frac{Ba\,\mathrm{sin}.\zeta^2 - Ba - Ab\,\mathrm{cos}.\zeta}{a\,\mathrm{sin}.\zeta^2} = \frac{-Ab\,\mathrm{cos}.\zeta - Ba\,\mathrm{cos}.\zeta^2}{a\,\mathrm{sin}.\zeta^2}:$$

ita vt reperiatur $D = -\frac{Ab - Ba\,\mathrm{cos}.\zeta}{2naab\,\mathrm{sin}.\zeta^2}$. Sumtis ergo litteris

$C = \frac{Ba + Ab\,\mathrm{cos}.\zeta}{2nabb\,\mathrm{sin}.\zeta^2}$ et $D = \frac{-Ab - Ba\,\mathrm{cos}.\zeta}{2naab\,\mathrm{sin}.\zeta^2}$, erit

$$y + \frac{C + Dx}{X^n} = \int \frac{-2nD\,\partial x}{X^n} + \int \frac{D\,\partial x}{X^n} = -(2n-1)D\int \frac{\partial x}{X^n}:$$

ideoque

$$\int \frac{(A + Bx)\,\partial x}{X^{n+1}} = \frac{-C - Dx}{X^n} - (2n-1)D\int \frac{\partial x}{X^n}, \text{ siue}$$

$$\int\frac{(A+Bx)\partial x}{X^{n+1}}=\frac{-Baa-Aab\cos.\zeta+(Abb+Bab\cos.\zeta)x}{2naabb\sin.\zeta^2 X^n}$$
$$+\frac{(2n-1)(Ab+Ba\cos.\zeta)}{2naab\sin.\zeta^2}\int\frac{\partial x}{X^n}.$$

Quare si formula $\int\frac{\partial x}{X^n}$ constet, etiam integrale hoc

$\int\frac{(A+Bx)\partial x}{X^{n+1}}$ assignari poterit.

Corollarium 1.

70. Cum igitur manente

$X=aa-2abx\cos.\zeta+bbxx$, fit

$\int\frac{\partial x}{X}=\frac{1}{ab\sin.\zeta}$ Arc. tang. $\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+$ Const. erit:

$$\int\frac{(A+Bx)\partial x}{X^2}=\frac{-Baa-Aab\cos.\zeta+(Abb+Bab\cos.\zeta)x}{2aabb\sin.\zeta^2 X}$$
$$+\frac{Ab+Ba\cos.\zeta}{2a^2bb\sin.\zeta^3}\text{ Arc. tang. }\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+\text{Const.}$$

Ideoque posito $B=0$ et $A=1$, fiet

$$\int\frac{\partial x}{X^2}=\frac{-a\cos.\zeta+bx}{2aab\sin.\zeta^2 X}+\frac{1}{2a^3b\sin.\zeta^3}\text{ Arc. tang. }\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+\text{Const.}$$

Integrale ergo $\int\frac{(A+Bx)\partial x}{X^2}$ logarithmos non inuoluit.

Corollarium 2.

71. Hinc ergo cum sit:

$$\int\frac{\partial x}{X^3}=\frac{-a\cos.\zeta+bx}{4aab\sin.\zeta^2.X^2}+\frac{3}{4aa\sin.\zeta^2}\int\frac{\partial x}{X^2}+\text{Const.}$$

erit illum valorem substituendo:

$$\int\frac{\partial x}{X^3}=\frac{-a\cos.\zeta+bx}{4aab\sin.\zeta^2.X^2}+\frac{3(-a\cos.\zeta+bx)}{2.4a^4b\sin.\zeta^4.X}$$
$$+\frac{1.3}{2.4a^5b\sin.\zeta^5}\text{ Arc. tang. }\frac{bx\sin.\zeta}{a-bx\cos.\zeta}.$$

Hincque porro concluditur:

$$\int \frac{\partial x}{X^4} = \frac{-a\cos.\zeta + bx}{6aab\sin.\zeta^2 . X^3} + \frac{5(-a\cos.\zeta + bx)}{4.6a^4 b\sin.\zeta^4 . X^2} + \frac{3.5(-a\cos.\zeta + bx)}{2.4.6a^6 b\sin.\zeta^6 . X}$$
$$+ \frac{1.3.5}{2.4.6a^7 b\sin.\zeta^7} \text{Arc. tang.} \frac{bx\sin.\zeta}{a - bx\cos.\zeta}.$$

Corollarium 3.

72. Sic vlterius progrediendo, omnium huiusmodi formularum integralia obtinebuntur:

$$\int \frac{\partial x}{X}, \int \frac{\partial x}{X^2}, \int \frac{\partial x}{X^3}, \int \frac{\partial x}{X^4}, \text{ etc.}$$

quorum primum arcu circulari solo exprimitur, reliqua vero praeterea partes algebraicas continent.

Scholion.

73. Sufficit autem integralia $\int \frac{\partial x}{X^{n+1}}$ nosse, quia formula $\int \frac{(A + Bx)\partial x}{X^{n+1}}$ facile eo reducitur: ita enim repraesentari potest:

$$\frac{1}{2bb} \int \frac{2Abb\partial x + 2Bbbx\partial x - 2Bab\partial x\cos.\zeta + 2Bab\partial x\cos.\zeta}{X^{n+1}},$$

quae ob $2bbx\partial x - 2ab\partial x\cos.\zeta = \partial X$, abit in hanc

$$\frac{1}{2bb} \int \frac{B\partial X}{X^{n+1}} + \frac{1}{b} \int \frac{(Ab + Ba\cos.\zeta)\partial x}{X^{n+1}}.$$

At $\int \frac{\partial X}{X^{n+1}} = -\frac{1}{nX^n}$, vnde habebitur:

$$\int \frac{(A + Bx)\partial x}{X^{n+1}} = \frac{-B}{2nbbX^n} + \frac{Ab + Ba\cos.\zeta}{b} \int \frac{\partial x}{X^{n+1}};$$

vnde tantum opus est nosse integralia $\int \frac{\partial X}{X^{n+1}}$, quae modo exhibuimus. Atque haec sunt omnia subsidia quibus indigemus ad omnes formulas fractas $\frac{M}{N}\partial x$ integrandas, dummodo M et

M et N sunt functiones integrae ipsius x. Quocirca in genere integratio omnium huiusmodi formularum $\int V \partial x$, vbi V est functio rationalis ipsius x quaecunque, est in potestate: de quibus notandum est, nisi integralia fuerint algebraica, semper vel per logarithmos vel angulos exhiberi posse. Nihil aliud igitur superest, nisi vt hanc methodum aliquot exemplis illustremus.

Exemplum 1.

74. *Proposita formula differentiali* $\frac{(A + Bx)\partial x}{\alpha + \beta x + \gamma x x}$, *definire eius integrale.*

Cum in numeratore variabilis x pauciores habeat dimensiones, quam in denominatore, haec fractio nullas partes integras complectitur. Hinc denominatoris indoles perpendatur, vtrum habeat duos factores simplices reales nec ne? ac priori casu num factores sint aequales? ex quo tres habebimus casus euoluendos.

I. Habeat denominator ambos factores aequales, sitque $= (a + bx)^2$, et fractio $\frac{A + Bx}{(a + bx)^2}$ resoluitur in duas,

$$\frac{Ab - Ba}{b(a + bx)^2} + \frac{B}{b(a + bx)}, \text{ vnde fit:}$$

$$\int \frac{(A + Bx)\partial x}{(a + bx)^2} = \frac{Ba - Ab}{bb(a + bx)} + \frac{B}{bb} l(a + bx) + \text{Const.}$$

Si integrale ita determinetur, vt euanescat posito $x = 0$, reperitur

$$\int \frac{(A + Bx)\partial x}{(a + bx)^2} = \frac{(Ab - Ba)x}{ab(a + bx)} + \frac{B}{bb} l \frac{a + bx}{a}.$$

II. Habeat denominator duos factores inaequales, sitque proposita haec formula $\frac{A + Bx}{(a + bx)(f + gx)} \partial x$, et haec fractio resoluitur in has partiales:

$$\frac{Ab - Ba}{bf - ag} \cdot \frac{\partial x}{a + bx} + \frac{Ag - Bf}{ag - bf} \cdot \frac{\partial x}{f + gx};$$

vnde obtinetur integrale quaesitum:

$$\int\frac{(A+Bx)\partial x}{(a+bx)(f+gx)}=\frac{Ab-BA}{b(bf-ag)}l\frac{a+bx}{a}+\frac{Ag-Bf}{g(ag-bf)}l\frac{f+gx}{f}+\text{Const.}$$

Ponatur:

$$\frac{Ab-Ba}{b(bf-ag)}=m+n \text{ et } \frac{Bf-Ag}{g(bf-ag)}=m-n,$$

vt integrale fiat:

$$m\,l\frac{(a+bx)(f+gx)}{af}+n\,l\frac{f(a+bx)}{a(f+gx)}, \text{ erit:}$$

$$2m=\frac{B(bf-ag)}{bg(bf-ag)}=\frac{B}{bg}, \text{ et}$$

$$2n=\frac{2Abg-Bag-Bbf}{bg(bf-ag)}. \text{ Erit ergo:}$$

$$\int\frac{(A+Bx)\partial x}{(a+bx)(f+gx)}=\frac{B}{2bg}l\frac{(a+bx)(f+gx)}{af}+\frac{2Abg-B(ag+bf)}{2bg(bf-ag)}l\frac{f(a+bx)}{a(f+gx)}.$$

III. Sint denominatoris factores simplices ambo imaginarii, quo casu formam habebit $aa-2abx\cos.\zeta+bbxx$; qui casus cum supra iam sit tractatus, erit:

$$\int\frac{(A+Bx)\partial x}{aa-2abx\cos.\zeta+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa-2abx\cos.\zeta+bbxx)}}{a}$$
$$+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc. tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}.$$

Corollarium 1.

75. Casu secundo, quo $f=a$, et $g=-b$, erit

$$\int\frac{(A+Bx)\partial x}{aa-bbxx}=\frac{-B}{2bb}l\frac{aa-bbxx}{aa}+\frac{A}{2ab}l\frac{a+bx}{a-bx}.$$

Hinc seorsim sequitur:

$$\int\frac{A\partial x}{aa-bbxx}=\frac{A}{2ab}l\frac{a+bx}{a-bx}+C \text{ et}$$

$$\int\frac{Bx\partial x}{aa-bbxx}=\frac{-B}{2bb}l\frac{aa-bbxx}{aa}=\frac{B}{bb}l\frac{a}{\sqrt{(aa-bbxx)}}+C.$$

Corollarium 2.

76. Casu tertio, si ponamus $\cos.\zeta=0$, habemus

$$\int\frac{(A+Bx)\partial x}{aa+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+bbxx)}}{a}+\frac{A}{ab}\text{Arc. tang.}\frac{bx}{a}+C,$$

hincque sigillatim:

$$\int\frac{A\partial x}{aa+bbxx}=\frac{A}{ab}\text{Arc. tang.}\frac{bx}{a}+C, \text{ et}$$

$$\int\frac{Bx\partial x}{aa+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+bbxx)}}{a}+C.$$

Ex-

Exemplum 2.

77. *Proposita formula differentiali* $\frac{x^{m-1}\partial x}{1+x^n}$, *siquidem exponens* m — 1 *minor sit quam* n, *integrale definire.*

In capite vltimo Institut. Calculi Differential. inuenimus fractiones simplices, in quas haec fractio $\frac{x^m}{1+x^n}$ resoluitur, sumto π pro mensura duorum angulorum rectorum, in hac forma generali contineri:

$$\frac{2\,\text{fin.}\frac{(2k-1)\pi}{n}\,\text{fin.}\frac{m(2k-1)\pi}{n} - 2\,\text{cof.}\frac{m(2k-1)\pi}{n}\left(x-\text{cof.}\frac{(2k-1)\pi}{n}\right)}{n\left(1-2x\,\text{cof.}\frac{(2k-1)\pi}{n}+xx\right)}:$$

vbi pro k successiue omnes numeros 1, 2, 3, etc. substitui conuenit, quoad $2k-1$ numerum n superare incipiat. Hac ergo forma in ∂x ducta, et cum generali nostra

$\frac{(A+Bx)\partial x}{aa-2axb\,\text{cof.}\zeta+bbxx}$ comparata, fit

$a=1$, $b=1$, $\zeta=\frac{(2k-1)\pi}{n}$; et

$$A=\tfrac{2}{n}\,\text{fin.}\tfrac{(2k-1)\pi}{n}\,\text{fin.}\tfrac{m(2k-1)\pi}{n}+\tfrac{2}{n}\,\text{cof.}\tfrac{(2k-1)\pi}{n}\,\text{cof.}\tfrac{m(2k-1)\pi}{n},$$

seu $A=\frac{2}{n}\,\text{cof.}\frac{(m-1)(2k-1)\pi}{n}$, et

$B=-\frac{2}{n}\,\text{cof.}\frac{m(2k-1)\pi}{n}$, vnde fit

$$Ab+Ba\,\text{cof.}\zeta=\tfrac{2}{n}\,\text{fin.}\tfrac{(2k-1)\pi}{n}\,\text{fin.}\tfrac{m(2k-1)\pi}{n}:$$

ac propterea huius partis integrale erit =

$$-\tfrac{2}{n}\,\text{cof.}\tfrac{m(2k-1)\pi}{n}\,l\sqrt{\left(1-2x\,\text{cof.}\tfrac{(2k-1)\pi}{n}+xx\right)}$$

$$+\tfrac{2}{n}\,\text{fin.}\tfrac{m(2k-1)\pi}{n}\,\text{Arc. tang.}\frac{x\,\text{fin.}\frac{(2k-1)\pi}{n}}{1-x\,\text{cof.}\frac{(2k-1)\pi}{n}}.$$

Ac si n sit numerus impar, praeterea accedit fractio $\frac{\pm\partial x}{n(1+x)}$, cuius integrale est $\pm\frac{1}{n}l(1+x)$: vbi signum superius valet, si

m

m impar, inferius vero, si m par. Quocirca integrale quaesitum $\int \frac{x^{m-1}\partial x}{1+x^n}$, sequenti modo exprimetur:

$$-\frac{2}{n}\cos.\frac{m\pi}{n}\,l\sqrt{(1-2x\cos.\frac{\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{m\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{\pi}{n}}{1-x\cos.\frac{\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{3m\pi}{n}\,l\sqrt{(1-2x\cos.\frac{3\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{3m\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{3\pi}{n}}{1-x\cos.\frac{3\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{5m\pi}{n}\,l\sqrt{(1-2x\cos.\frac{5\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{5m\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{5\pi}{n}}{1-x\cos.\frac{5\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{7m\pi}{n}\,l\sqrt{(1-2x\cos.\frac{7\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{7m\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{7\pi}{n}}{1-x\cos.\frac{7\pi}{n}}$$

etc.

secundum numeros impares ipso n minores, sicque totum obtinetur integrale si n fuerit numerus par; sin autem n sit numerus impar, insuper accedit haec pars $\pm\frac{1}{n}\,l\,(1+x)$, prout m sit numerus vel impar vel par: vnde si $m=1$, accedit insuper $+\frac{1}{n}\,l\,(1+x)$.

Corollarium 1.

78. Sumamus $m=1$, vt habeatur forma $\int\frac{\partial x}{1+x^n}$, et pro variis casibus ipsius n adipiscimur:

I. $\int\frac{\partial x}{1+x}=l\,(1+x)$

II. $\int\frac{\partial x}{1+x^2}=\text{Arc. tang. } x$

III. $\int\frac{\partial x}{1+x^3}=-\frac{2}{3}\cos.\frac{\pi}{3}\,l\sqrt{(1-2x\cos.\frac{\pi}{3}+xx)}+\frac{2}{3}\sin.\frac{\pi}{3}\,\text{Arc.tang.}\frac{x\sin.\frac{\pi}{3}}{1-x\cos.\frac{\pi}{3}}$

$+\frac{1}{3}\,l\,(1+x)$

IV.

$$\text{IV.} \int \frac{\partial x}{1+x^4} = \begin{cases} -\frac{2}{4}\cos.\frac{\pi}{4}\, l\sqrt{(1-2x\cos.\frac{\pi}{4}+xx)} + \frac{2}{4}\sin.\frac{\pi}{4}\,\text{Arc.tang.}\frac{x\sin.\frac{\pi}{4}}{1-x\cos.\frac{\pi}{4}} \\ -\frac{2}{4}\cos.\frac{3\pi}{4}\, l\sqrt{(1-2x\cos.\frac{3\pi}{4}+xx)} + \frac{2}{4}\sin.\frac{3\pi}{4}\,\text{Arc.tang.}\frac{x\sin.\frac{3\pi}{4}}{1-x\cos.\frac{3\pi}{4}} \end{cases}$$

$$\text{V.} \int \frac{\partial x}{1+x^5} = \begin{cases} -\frac{2}{5}\cos.\frac{\pi}{5}\, l\sqrt{(1-2x\cos.\frac{\pi}{5}+xx)} + \frac{2}{5}\sin.\frac{\pi}{5}\,\text{Arc.tang.}\frac{x\sin.\frac{\pi}{5}}{1-x\cos.\frac{\pi}{5}} \\ -\frac{2}{5}\cos.\frac{3\pi}{5}\, l\sqrt{(1-2x\cos.\frac{3\pi}{5}+xx)} + \frac{2}{5}\sin.\frac{3\pi}{5}\,\text{Arc.tang.}\frac{x\sin.\frac{3\pi}{5}}{1-x\cos.\frac{3\pi}{5}} \\ +\frac{1}{5}\, l\,(1+x) \end{cases}$$

$$\text{VI.} \int \frac{\partial x}{1+x^6} = \begin{cases} -\frac{2}{6}\cos.\frac{\pi}{6}\, l\sqrt{(1-2x\cos.\frac{\pi}{6}+xx)} + \frac{2}{6}\sin.\frac{\pi}{6}\,\text{Arc.tang.}\frac{x\sin.\frac{\pi}{6}}{1-x\cos.\frac{\pi}{6}} \\ -\frac{2}{6}\cos.\frac{3\pi}{6}\, l\sqrt{(1-2x\cos.\frac{3\pi}{6}+xx)} + \frac{2}{6}\sin.\frac{3\pi}{6}\,\text{Arc.tang.}\frac{x\sin.\frac{3\pi}{6}}{1-x\cos.\frac{3\pi}{6}} \\ -\frac{2}{6}\cos.\frac{5\pi}{6}\, l\sqrt{(1-2x\cos.\frac{5\pi}{6}+xx)} + \frac{2}{6}\sin.\frac{5\pi}{6}\,\text{Arc.tang.}\frac{x\sin.\frac{5\pi}{6}}{1-x\cos.\frac{5\pi}{6}} \end{cases}$$

Corollarium 2.

79. Loco finuum et cofinuum valores, vbi commode fieri poteft, fubftituendo, obtinemus:

$$\int \frac{\partial x}{1+x^3} = -\frac{1}{3}\, l\sqrt{(1-x+xx)} + \frac{1}{\sqrt{3}}\,\text{Arc.tang.}\frac{x\sqrt{3}}{2-x} + \frac{1}{3}\, l\,(1+x)$$

feu

$$\int \frac{\partial x}{1+x^3} = \frac{1}{3}\, l\, \frac{1+x}{\sqrt{(1-x+xx)}} + \frac{1}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x}.$$

Deinde ob $\sin.\frac{\pi}{4} = \cos.\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \sin.\frac{3\pi}{4} = -\cos.\frac{3\pi}{4}$, fit

$$\int \frac{\partial x}{1+x^4} = +\frac{1}{2\sqrt{2}}\, l\, \frac{\sqrt{(1+x\sqrt{2}+xx)}}{\sqrt{(1-x\sqrt{2}+xx)}} + \frac{1}{2\sqrt{2}}\,\text{Arc. tang.}\frac{x\sqrt{2}}{1-xx},$$

tum vero

$$\int \frac{\partial x}{1+x^6} = \frac{1}{2\sqrt{3}}\, l\, \frac{\sqrt{(1+x\sqrt{3}+xx)}}{\sqrt{(1-x\sqrt{3}+xx)}} + \frac{1}{6}\,\text{Arc. tang.}\frac{3x(1-xx)}{1-4xx+x^4}.$$

 Ex-

Exemplum 3.

80. *Proposita formula differentiali* $\frac{x^{m-1}\,dx}{1-x^n}$, *siquidem exponens* m − 1 *sit minor quam* n, *eius integrale definire.*

Functionis fractae $\frac{x^{m-1}}{1-x^n}$ pars, ex factore quocunque oriunda, hac forma continetur:

$$\frac{2\,\text{fin.}\frac{2k\pi}{n}\,\text{fin.}\frac{2mk\pi}{n} - \text{cof.}\frac{2mk\pi}{n}\left(x - \text{cof.}\frac{2k\pi}{n}\right)}{n\left(1 - 2x\,\text{cof.}\frac{2k\pi}{n} + xx\right)}$$

quae cum forma nostra $\frac{A + Bx}{aa - 2abx\,\text{cof.}\,\zeta + bbxx}$ comparata, dat $a = 1$, $b = 1$, $\zeta = \frac{2k\pi}{n}$;

$$A = \tfrac{2}{n}\,\text{fin.}\tfrac{2k\pi}{n}\,\text{fin.}\tfrac{2mk\pi}{n} + \tfrac{2}{n}\,\text{cof.}\tfrac{2k\pi}{n}\,\text{cof.}\tfrac{2mk\pi}{n},$$

$$B = -\tfrac{2}{n}\,\text{cof.}\tfrac{2mk\pi}{n}:\ \text{hincque}$$

$$Ab + Ba\,\text{cof.}\,\zeta = \tfrac{2}{n}\,\text{fin.}\tfrac{2k\pi}{n}\,\text{fin.}\tfrac{2mk\pi}{n}.$$

Ex quo integrale hinc oriundum erit =

$$-\tfrac{2}{n}\,\text{cof.}\tfrac{2km\pi}{n}\,l\sqrt{\left(1 - 2x\,\text{cof.}\tfrac{2k\pi}{n} + xx\right)}$$

$$+\tfrac{2}{n}\,\text{fi.}\tfrac{2km\pi}{n}\,\text{Arc. tang.}\,\frac{x\,\text{fin.}\frac{2k\pi}{n}}{1 - x\,\text{cof.}\frac{2k\pi}{n}}:$$

vbi pro k successiue omnes numeri 0, 1, 2, 3, etc. substitui debent, quamdiu $2k$ non superat n. At casu $k = 0$ fit integralis pars $= -\frac{1}{n}\,l(1-x)$: et quando n est numerus par, vltima pars oritur ex $2k = n$, quae ergo erit

$$-\tfrac{2}{n}\,\text{cof.}\,m\pi\,l\sqrt{(1 + 2x + xx)} = -\tfrac{\text{cof.}\,m\pi}{n}\,l(1+x):$$

ergo si m est par, erit cof. $m\pi = +1$, at si m impar, fit cof. $m\pi = -1$. Quocirca integrale $\int\frac{x^{m-1}\,\partial x}{1-x^n}$, hoc modo

ex-

exprimitur:

$$-\tfrac{1}{n}\, l\,(1-x)$$

$$-\tfrac{2}{n}\cos.\tfrac{2m\pi}{n}\, l\sqrt{(1-2x\cos.\tfrac{2\pi}{n}+xx)}$$

$$+\tfrac{2}{n}\sin.\tfrac{2m\pi}{n}\ \text{Arc. tang.}\ \frac{x\sin.\frac{2\pi}{n}}{1-x\cos.\frac{2\pi}{n}}$$

$$-\tfrac{2}{n}\cos.\tfrac{4m\pi}{n}\, l\sqrt{(1-2x\cos.\tfrac{4\pi}{n}+xx)}$$

$$+\tfrac{2}{n}\sin.\tfrac{4m\pi}{n}\ \text{Arc. tang.}\ \frac{x\sin.\frac{4\pi}{n}}{1-x\cos.\frac{4\pi}{n}}$$

$$-\tfrac{2}{n}\cos.\tfrac{6m\pi}{n}\, l\sqrt{(1-2x\cos.\tfrac{6\pi}{n}+xx)}$$

$$+\tfrac{2}{n}\sin.\tfrac{6m\pi}{n}\ \text{Arc. tang.}\ \frac{x\sin.\frac{6\pi}{n}}{1-x\cos.\frac{6\pi}{n}},\ \text{etc.}$$

Corollarium.

81. Sit $m=1$, et pro n successiue numeri 1, 2, 3, etc. substituantur, vt nanciscamur sequentes integrationes:

$$\text{I. } \int\frac{\partial x}{1-x}=-l\,(1-x)$$

$$\text{II. } \int\frac{\partial x}{1-xx}=-\tfrac{1}{2}l(1-x)+\tfrac{1}{2}l\,(1+x)=\tfrac{1}{2}l\,\frac{1+x}{1-x}$$

$$\text{III. } \int\frac{\partial x}{1-x^3}=\begin{cases}-\tfrac{1}{3}l\,(1-x)-\tfrac{2}{3}\cos.\tfrac{2}{3}\pi\, l\sqrt{(1-2x\cos.\tfrac{2}{3}\pi+xx)}\\ \qquad +\tfrac{2}{3}\sin.\tfrac{2}{3}\pi\ \text{Arc. tang.}\ \dfrac{x\sin.\frac{2}{3}\pi}{1-x\cos.\frac{2}{3}\pi}\end{cases}$$

$$\text{IV. } \int\frac{\partial x}{1-x^4}=\begin{cases}-\tfrac{1}{4}l\,(1-x)-\tfrac{2}{4}\cos.\tfrac{2}{4}\pi\, l\sqrt{(1-2x\cos.\tfrac{2}{4}\pi+xx)}\\ \qquad +\tfrac{2}{4}\sin.\tfrac{2}{4}\pi\ \text{Arc. tang.}\ \dfrac{x\sin.\frac{2}{4}\pi}{1-x\cos.\frac{2}{4}\pi}.\\ +\tfrac{1}{4}l\,(1+x)\end{cases}$$

$$\text{V. } \int\frac{\partial x}{1-x^5}=\begin{cases}-\tfrac{1}{5}l\,(1-x)-\tfrac{2}{5}\cos.\tfrac{2}{5}\pi\, l\sqrt{(1-2x\cos.\tfrac{2}{5}\pi+xx)}\\ \qquad +\tfrac{2}{5}\sin.\tfrac{2}{5}\pi\ \text{Arc. tang.}\ \dfrac{x\sin.\frac{2}{5}\pi}{1-x\cos.\frac{2}{5}\pi}\\ \qquad -\tfrac{2}{5}\cos.\tfrac{4}{5}\pi\, l\sqrt{(1-2x\cos.\tfrac{4}{5}\pi+xx)}\\ \qquad +\tfrac{2}{5}\sin.\tfrac{4}{5}\pi\ \text{Arc. tang.}\ \dfrac{x\sin.\frac{4}{5}}{1-x\cos.\frac{4}{5}\pi}\end{cases}$$

$$\text{VI. } \int \frac{\partial x}{1-x^6} = \begin{cases} -\frac{1}{6} l(1-x) - \frac{2}{6} \operatorname{cos.} \frac{2}{6}\pi \, l \sqrt{(1-2x \operatorname{cos.} \frac{2}{6}\pi + xx)} \\ \qquad + \frac{2}{6} \operatorname{fin.} \frac{2}{6}\pi \text{ Arc. tang. } \dfrac{x \operatorname{fin.} \frac{2}{6}\pi}{1-x \operatorname{cof.} \frac{2}{6}\pi} \\ + \frac{1}{6} l(1+x) - \frac{2}{6} \operatorname{cof.} \frac{4}{6}\pi \, l \sqrt{(1-2x \operatorname{cof.} \frac{4}{6}\pi + xx)} \\ \qquad + \frac{2}{6} \operatorname{fin.} \frac{4}{6}\pi \text{ Arc. tang. } \dfrac{x \operatorname{fin.} \frac{4}{6}\pi}{1-x \operatorname{cof.} \frac{4}{6}\pi}. \end{cases}$$

Exemplum 4.

82. *Propofita formula differentiali* $\frac{(x^{m-1} + x^{n-m-1})\partial x}{1+x^n}$, *exiftente* n > m − 1, *eius integrale definire.*

Ex exemplo 2do patet, integralis partem quamcunque in genere effe, fumto i pro numero quocunque impare non maiore quam n,

$$-\frac{2}{n} \operatorname{cof.} \frac{im\pi}{n} \, l \sqrt{(1 - 2x \operatorname{cof.} \frac{i\pi}{n} + xx)}$$

$$+\frac{2}{n} \operatorname{fin.} \frac{im\pi}{n} \text{ Arc. tang. } \frac{x \operatorname{fin.} \frac{i\pi}{n}}{1 - x \operatorname{cof.} \frac{i\pi}{n}}$$

$$-\frac{2}{n} \operatorname{cof.} \frac{i(n-m)\pi}{n} \, l \sqrt{(1 - 2x \operatorname{cof.} \frac{i\pi}{n} + xx)}$$

$$+\frac{2}{n} \operatorname{fin.} \frac{i(n-m)\pi}{n} \text{ Arc. tang. } \frac{x \operatorname{fin.} \frac{i\pi}{n}}{1 - x \operatorname{cof.} \frac{i\pi}{n}}.$$

Verum eft

$\operatorname{cof.} \frac{i(n-m)\pi}{n} = \operatorname{cof.} (i\pi - \frac{im\pi}{n}) = -\operatorname{cof.} \frac{im\pi}{n}$, et

$\operatorname{fin.} \frac{i(n-m)\pi}{n} = \operatorname{fin.} (i\pi - \frac{im\pi}{n}) = +\operatorname{fin.} \frac{im\pi}{n}$:

vnde partes logarithmicae fe deftruent, eritque pars integralis in genere,

$$+\frac{4}{n} \operatorname{fin.} \frac{im\pi}{n} \text{ Arc. tang. } \frac{x \operatorname{fin.} \frac{i\pi}{n}}{1 - x \operatorname{cof.} \frac{i\pi}{n}}.$$

Po-

Ponatur commoditatis ergo angulus $\frac{\pi}{n} = \omega$, eritque

$$\int \frac{(x^{m-1} + x^{n-m-1})\,\partial x}{1 + x^n} = + \frac{4}{n} \text{ fin. } m\omega \text{ Arc. tang. } \frac{x \text{ fin. } \omega}{1 - x \text{ cof. } \omega}$$

$$+ \frac{4}{n} \text{ fin. } 3m\omega \text{ Arc. tang. } \frac{x \text{ fin. } 3\omega}{1 - x \text{ cof. } 3\omega}$$

$$+ \frac{4}{n} \text{ fin. } 5m\omega \text{ Arc. tang. } \frac{x \text{ fin. } 5\omega}{1 - x \text{ cof. } 5\omega}$$

$$\vdots$$

$$+ \frac{4}{n} \text{ fin. } im\omega \text{ Arc. tang. } \frac{x \text{ fin. } i\omega}{1 - x \text{ cof. } i\omega}:$$

fumto pro i maximo numero impare, exponentem n non excedente. Si ipfe numerus n fit impar, pars ex pofitione $i = n$ oriunda, ob fin. $m\pi = 0$, euanefcet. Notetur ergo, hic totum integrale per meros angulos exprimi.

Corollarium.

83. Simili modo fequens integrale elicitur, vbi foli logarithmi relinquentur, manente $\frac{\pi}{n} = \omega$:

$$\int \frac{(x^{m-1} - x^{n-m-1})\,\partial x}{1 + x^n} = - \frac{4}{n} \text{ cof. } m\omega\, l\sqrt{(1 - 2x \text{ cof. } \omega + xx)}$$

$$- \frac{4}{n} \text{ cof. } 3m\omega\, l\sqrt{(1 - 2x \text{ cof. } 3\omega + xx)}$$

$$- \frac{4}{n} \text{ cof. } 5m\omega\, l\sqrt{(1 - 2x \text{ cof. } 5\omega + xx)}$$

$$\vdots$$

$$- \frac{4}{n} \text{ cof. } im\omega\, l\sqrt{(1 - 2x \text{ cof. } i\omega + xx)}:$$

donec fcilicet numerus impar i non fuperet exponentem n.

Exemplum 5.

84. *Propofita formula differentiali* $\frac{(x^{m-1} - {}^{n-m-1})\,\partial x}{1 - x^n}$, *exiftente* $n > m - 1$, *eius integrale definire.*

Ex exemplo 3^{tio} integralis pars quaecunque concluditur, fiquidem breuitatis gratia $\frac{\pi}{n} = \omega$ ftatuamus:

$$- \tfrac{2}{n} \text{cof.}\, 2\,k\,m\,\omega\, l \sqrt{(1 - 2\,x\, \text{cof.}\, 2\,k\,\omega + x\,x)}$$
$$+ \tfrac{2}{n} \text{fin.}\, 2\,k\,m\,\omega\, \text{Arc. tang.}\, \frac{x\, \text{fin.}\, 2\,k\,\omega}{1 - x\, \text{cof.}\, 2\,k\,\omega}$$
$$+ \tfrac{2}{n} \text{cof.}\, 2\,k\,(n-m)\,\omega\, l \sqrt{(1 - 2\,x\, \text{cof.}\, 2\,k\,\omega + x\,x)}$$
$$- \tfrac{2}{n} \text{fin.}\, 2\,k\,(n-m)\,\omega\, \text{Arc. tang.}\, \frac{x\, \text{fin.}\, 2\,k\,\omega}{1 - x\, \text{cof.}\, 2\,k\,\omega}.$$

At eft:

$$\text{cof.}\, 2\,k\,(n-m)\,\omega = \text{cof.}\,(2\,k\,\pi - 2\,k\,m\,\omega) = \text{cof.}\, 2\,k\,m\,\omega, \text{ et}$$
$$\text{fin.}\, 2\,k\,(n-m)\,\omega = \text{fin.}\,(2\,k\,\pi - 2\,k\,m\,\omega) = - \text{fin.}\, 2\,k\,m\,\omega;$$

vnde ifta pars generalis abit in: $\frac{4}{n} \text{fin.}\, 2\,k\,m\,\omega\, \text{Arc. tang.}\, \frac{x\, \text{fin.}\, 2\,k\,\omega}{1 - x\, \text{cof.}\, 2\,k\,\omega}$.
Quare hinc ifta integratio colligitur:

$$\int \frac{(x^{m-1} + x^{n-m-1})\,\partial x}{1 - x^n} = + \tfrac{4}{n} \text{fin.}\, 2\,m\,\omega\, \text{Arc. tang.}\, \frac{x\, \text{fin.}\, 2\,\omega}{1 - x\, \text{cof.}\, 2\,\omega}$$
$$+ \tfrac{4}{n} \text{fin.}\, 4\,m\,\omega\, \text{Arc. tang.}\, \frac{x\, \text{fin.}\, 4\,\omega}{1 - x\, \text{cof.}\, 4\,\omega}$$
$$+ \tfrac{4}{n} \text{fin.}\, 6\,m\,\omega\, \text{Arc. tang.}\, \frac{x\, \text{fin.}\, 6\,\omega}{1 - x\, \text{cof.}\, 6\,\omega}$$
$$\text{etc.}$$

numeris paribus tamdiu afcendendo, quoad exponentem n non fuperent.

Corollarium.

85. Indidem etiam haec integratio abfoluitur, manente $\frac{\pi}{n} = \omega$:

$$\int \frac{(x^{m-1} + x^{n-m-1})\,\partial x}{1 - x^n} = - \tfrac{2}{n}\, l\,(1 - x)$$
$$- \tfrac{4}{n} \text{cof.}\, 2\,m\,\omega\, l \sqrt{(1 - 2\,x\, \text{cof.}\, 2\,\omega + x\,x)}$$
$$- \tfrac{4}{n} \text{cof.}\, 4\,m\,\omega\, l \sqrt{(1 - 2\,x\, \text{cof.}\, 4\,\omega + x\,x)}$$
$$- \tfrac{4}{n} \text{cof.}\, 6\,m\,\omega\, l \sqrt{(1 - 2\,x\, \text{cof.}\, 6\,\omega + x\,x)}$$
$$\text{etc.}$$

vbi

vbi etiam numeri pares non vltra terminum n sunt continuandi.

Exemplum 6.

86. *Proposita formula differentiali* $\partial y = \frac{\partial x}{x^3(1+x)(1-x^4)}$, *eius integrale inuenire.*

Functio fracta per ∂x affecta secundum denominatoris factores est $\frac{1}{x^3(1+x)^2(1-x)(1+xx)}$, quae in has fractiones simplices resoluitur:

$$\frac{1}{x^3} - \frac{1}{x^2} + \frac{1}{x} - \frac{1}{4(1+x)^2} - \frac{9}{8(1+x)} + \frac{1}{8(1-x)} + \frac{1+x}{4(1+xx)} = \frac{\partial y}{\partial x};$$

vnde per integrationem elicitur:

$$y = -\frac{1}{2x^2} + \frac{1}{x} + lx + \frac{1}{4(1+x)} - \frac{9}{8}l(1+x) - \frac{1}{8}l(1-x)$$
$$+ \frac{1}{8}l(1+xx) + \frac{1}{4}\,\text{Arc. tang. } x,$$

quae expressio in hanc formam transmutatur

$$y = C \frac{-2+2x+5xx}{4xx(1+x)} - l\frac{1+x}{x} + \frac{1}{8}l\frac{1+xx}{1-xx} + \frac{1}{4}\,\text{Arc. tang. } x.$$

Scholion.

87. Hoc igitur caput ita pertractare licuit, vt nihil amplius in hoc genere desiderari possit. Quoties ergo eiusmodi functio y ipsius x quaeritur, vt $\frac{\partial y}{\partial x}$ aequetur functioni rationali ipsius x, toties integratio nihil habet difficultatis, nisi forte ad denominatoris singulos factores eliciendos Algebrae praecepta non sufficiant: verum tum defectus ipsi Algebrae, non vero methodo integrandi, quam hic tractamus, est tribuendus. Deinde etiam potissimum notari conuenit, semper, cum $\frac{\partial y}{\partial x}$ functioni rationali ipsius x aequale ponitur, functionem y, nisi sit algebraica, alias quantitates transcendentes non inuoluere praeter logarithmos et angulos: vbi quidem obseruandum est, hic perpetuo logarithmos hyperbolicos intelli-

gi

gi oportere, cum ipſius $l\,x$ differentiale non ſit $= \frac{\partial x}{x}$, niſi logarithmus hyperbolicus ſumatur: at horum reductio ad vulgares eſt facillima, ita vt hinc applicatio calculi ad praxin nulli impedimento ſit obnoxia. Quare progrediamur ad eos caſus, quibus formula $\frac{\partial y}{\partial x}$ functioni irrationali ipſius x aequatur, vbi quidem primo notandum eſt, quoties iſta functio per idoneam ſubſtitutionem ad rationalitatem perduci poterit, caſum ad hoc caput reuolui. Veluti ſi fuerit $\partial y = \frac{(1+\sqrt{x}-\sqrt[3]{xx})}{1+\sqrt[3]{x}}\partial x$, euidens eſt, ponendo $x = z^6$, vnde fit $\partial x = 6\,z^5\,\partial z$, fore

$$\partial y = \frac{(1+z^3-z^4)}{1+zz}\,.\,6\,z^5\,\partial z\text{, ideoque}$$

$$\frac{\partial y}{\partial z} = -\,6z^7 + 6z^6 + 6z^5 - 6z^4 + 6zz - 6 + \frac{6}{1+zz},$$

vnde integrale

$$y = -\tfrac{3}{4}z^8 + \tfrac{6}{7}z^7 + z^6 - \tfrac{6}{5}z^5 + 2z^3 - 6z + 6\text{ Arc. tang. } z,$$

et reſtituto valore

$$y = -\tfrac{3}{4}x\sqrt[3]{x} + \tfrac{6}{7}x\sqrt[6]{x} + x - \tfrac{6}{5}\sqrt[6]{x^5} + 2\sqrt{x} - 6\sqrt[6]{x}$$
$$+\,6\text{ Arc. tang. }\sqrt[6]{x} + C.$$

CA-

CAPVT II.

DE INTEGRATIONE FORMVLARVM DIFFERENTIALIVM IRRATIONALIVM.

Problema 6.

88.

Propofita formula differentiali $\partial y = \frac{\partial x}{\sqrt{(\alpha + \beta x + \gamma x x)}}$, eius integrale inuenire.

Solutio.

Quantitas $\alpha x + \beta x + \gamma x x$, vel habet duos factores reales vel fecus.

I. Priori cafu formula propofita erit huiusmodi $\partial y = \frac{\partial x}{\sqrt{(a + b x)(f + g x)}}$. Statuatur ad irrationalitatem tollendam

$$(a + b x)\ (f + g x) = (a + b x)^2 z z,$$

erit $x = \frac{f - a z z}{b z z - g}$, ideoque

$$\partial x = \frac{2(a g - b f) z \partial z}{(b z z - g)^2} \text{ et } \sqrt{(a + b x)(f + g x)} = -\frac{(a g - b f) z}{b z z - g}:$$

vnde fit $\partial y = \frac{-2 \partial z}{b z z - g} = \frac{2 \partial z}{g - b z z}$, atque $z = \sqrt{\frac{f + g x}{a + b x}}$. Quare fi litterae b et g paribus fignis funt affectae, integrale per logarithmos, fin autem fignis difparibus, per angulos exprimetur.

II. Pofteriori cafu habebimus $\partial y = \frac{\partial x}{\sqrt{(a a - 2 a b x \cos. \zeta + b b x x)}}$. Statuatur

$$b b x x - 2 a b x \cos. \zeta + a a = (b x - a z)^2, \text{ erit}$$

$$-2 b x \cos. \zeta + a = -2 b x z + a z z \text{ et } x = \frac{a(1 - z z)}{2 b(\cos. \zeta - z)};$$

hinc

hinc $\partial x = \frac{a \partial z (1 - 2z \cos.\zeta + zz)}{2b(\cos.\zeta - z)^2}$, et

$$\sqrt{(aa - 2abx \cos.\zeta + bbxx)} = \frac{a(1 - 2z\cos.\zeta + zz)}{2(\cos.\zeta - z)}: \text{ ergo}$$

$$\partial y = \frac{\partial z}{b(\cos.\zeta - z)}, \text{ et } y = -\tfrac{1}{b}\, l\,(\cos.\zeta - z).$$

At est

$$z = \frac{bx - \sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a}, \text{ ideoque}$$

$$y = -\tfrac{1}{b}\, l\, \frac{a\cos.\zeta - bx + \sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a}, \text{ vel}$$

$$y = \tfrac{1}{b}\, l\,[-a\cos.\zeta + bx + \sqrt{(aa - 2abx\cos.\zeta + bbxx)}] + C.$$

Corollarium 1.

89. Casus vltimus latius patet, et ad formulam $\partial y = \frac{\partial x}{\sqrt{(\alpha + \beta x + \gamma xx)}}$, accomodari potest, dummodo fuerit γ quantitas positiua: namque ob $b = \sqrt{\gamma}$ et $a \cos.\zeta = \frac{-\beta}{2\sqrt{\gamma}}$, oritur,

$$y = \tfrac{1}{\sqrt{\gamma}}\, l\,[\tfrac{\beta}{2\sqrt{\gamma}} + x\sqrt{\gamma} + \sqrt{(\alpha + \beta x + \gamma xx)}] + C \text{ seu}$$

$$y = \tfrac{1}{\sqrt{\gamma}}\, l\,[\tfrac{1}{2}\beta + \gamma x + \sqrt{\gamma(\alpha + \beta x + \gamma xx)}] + C.$$

Corollarium 2.

90. Pro casu priori cum sit

$$\int \frac{2\partial z}{g - bzz} = \frac{1}{\sqrt{bg}}\, l\, \frac{\sqrt{g} + z\sqrt{b}}{\sqrt{g} - z\sqrt{b}} \text{ et}$$

$$\int \frac{2\partial z}{g + bzz} = \frac{2}{\sqrt{gb}} \text{ Arc. tang. } \frac{z\sqrt{b}}{\sqrt{g}},$$

habebimus hos casus:

$$\int \frac{\partial x}{\sqrt{(a+bx)(f+gx)}} = \frac{1}{\sqrt{bg}}\, l\, \frac{\sqrt{g(a+bx)} + \sqrt{b(f+gx)}}{\sqrt{g(a+bx)} - \sqrt{b(f+gx)}} + C$$

$$\int \frac{\partial x}{\sqrt{(bx-a)(f+gx)}} = \frac{1}{\sqrt{bg}}\, l\, \frac{\sqrt{g(bx-a)} + \sqrt{b(f+gx)}}{\sqrt{g(bx-a)} - \sqrt{b(f+gx)}} + C$$

$$\int \frac{\partial x}{\sqrt{(bx-a)(gx-f)}} = \frac{1}{\sqrt{bg}}\, l\, \frac{\sqrt{g(bx-a)} + \sqrt{b(gx-f)}}{\sqrt{g(bx-a)} - \sqrt{b(gx-f)}} + C$$

$$\int \frac{\partial x}{\sqrt{(a-bx)(f-gx)}} = \frac{1}{\sqrt{bg}}\, l\, \frac{\sqrt{g(a-bx)} + \sqrt{b(f-gx)}}{\sqrt{g(a-bx)} - \sqrt{b(f-gx)}} + C$$

$$\int \frac{\partial x}{\sqrt{(a-bx)(f+gx)}} = \frac{2}{\sqrt{bg}} \text{ Arc. tang. } \frac{\sqrt{b(f+gx)}}{\sqrt{g(a-bx)}} + C$$

$$\int \frac{\partial x}{\sqrt{(a-bx)(gx-f)}} = \frac{2}{\sqrt{bg}} \text{ Arc. tang. } \frac{\sqrt{b(gx-f)}}{\sqrt{g(a-bx)}} + C.$$

Corollarium 3.

91. Harum fex integrationum quatuor priores omnes in cafu Coroll. 1. continentur, binae autem poftremae in hac formula $\partial y = \frac{\partial}{\sqrt{(\alpha+\beta x-\gamma x x)}}$, continentur: fit enim pro penultima

$$af = \alpha,\ ag - bf = \beta,\ bg = \gamma,$$

vnde colligitur

$$y = \frac{1}{\sqrt{\gamma}} \text{ Arc. tang. } \frac{2\sqrt{\gamma(\alpha+\beta x-\gamma x x)}}{\beta - 2\gamma x};$$

fi fcilicet ille arcus duplicetur. Per cofinum autem erit

$$y = \frac{1}{\sqrt{\gamma}} \text{ Arc. cof. } \frac{\beta - 2\gamma x}{\sqrt{(\beta\beta+4\alpha\gamma)}} + C;$$

cuius veritas ex differentiatione patet.

Scholion 1.

92. Ex folutione huius problematis patet etiam, hanc formulam latius patentem $\frac{X\partial x}{\sqrt{(\alpha+\beta x+\gamma x x)}}$, fi X fuerit functio rationalis quaecunque ipfius x, per praecepta capitis praecedentis integrari poffe. Introducta enim loco x variabili z, qua formula radicalis rationalis redditur, etiam X abibit in functionem rationalem ipfius z. Idem adhuc generalius locum habet, fi pofito $\sqrt{(\alpha+\beta x+\gamma x x)} = u$, fuerit X functio quaecunque rationalis binarum quantitatum x et u, tum enim per fubftitutionem adhibitam, quia tam pro x quam pro u formulae rationales ipfius z fcribuntur, prodibit formula differentialis rationalis. Hoc idem etiam ita enunciari poteft, vt dicamus, formulae $X\partial x$, fi functio X nullam aliam irrationalem praeter $\sqrt{(\alpha+\beta x+\gamma x x)}$ inuoluat, integrale affignari poffe, propterea quod ea, ope fubftitutionis, in formulam differentialem rationalem transformari poteft.

Scholion 2.

93. Propofita autem formula differentiali quacunque irrationali, ante omnia videndum eft, num ea ope cuiuspiam fubftitutionis in rationalem transformari poffit? quod fi fuccedat, integratio per praecepta capitis praecedentis abfolui poterit: vnde fimul intelligitur, integrale nifi fit algebraicum, alias quantitates tranfcendentes non inuoluere praeter logarithmos et angulos. Quodfi autem nulla fubftitutio ad hoc idonea inueniri poffit, ab integrationis labore eft defiftendum, quandoquidem integrale neque algebraice neque per logarithmos vel angulos exprimere valemus. Veluti fi $X\partial x$ fuerit eiusmodi formula differentialis, quae nullo pacto ad rationalitatem reduci queat, eius integrale $\int X\partial x$, ad nouum genus functionum tranfcendentium erit referendum, in quo nihil aliud nobis relinquitur, nifi vt eius valorem vero proxime affignare conemur. Admiffo autem nouo genere quantitatum tranfcendentium, innumerabiles aliae formulae eo reduci atque integrari poterunt. Imprimis igitur in hoc erit elaborandum vt pro quolibet genere formula fimpliciffima notetur, qua conceffa reliquarum formularum integralia definire liceat. Hinc deducimur ad quaeftionem maximi momenti, quomodo integrationem formularum magis complicatarum ad fimpliciores reduci oporteat. Quod antequam aggrediamur, alias eiusmodi formulas perpendamus, quae ope idoneae fubftitutionis ab irrationalitate liberari queant; quemadmodum iam oftendimus, quoties X fuerit functio rationalis quantitatum

$$x \text{ et } u = \sqrt{(\alpha + \beta x + \gamma x x)},$$

ita vt alia irrationalitas non ingrediatur praeter radicem quadratam huiusmodi formulae $\alpha + \beta x + \gamma x x$, toties formulam differentialem $X\partial x$ in rationalem transformari poffe.

Pro-

Problema 7.

94. Propoſita formula differentiali $X\partial x(a+bx)^{\frac{\mu}{\nu}}$, in qua X denotet functionem quamcunque rationalem ipſius x, eam ab irrationalitate liberare.

Solutio.

Statuatur $a+bx=z^{\nu}$, vt fiat $(a+bx)^{\frac{\mu}{\nu}}=z^{\mu}$: tum quia $x=\frac{z^{\nu}-a}{b}$, facta hac ſubſtitutione, functio X abibit in functionem rationalem ipſius z, quae ſit Z, et ob $\partial x=\frac{\nu}{b}z^{\nu-1}\partial z$, formula noſtra differentialis induet hanc formam $\frac{\nu}{b}Zz^{\mu+\nu-1}\partial z$, quae cum ſit rationalis, per caput ſuperius integrari poteſt, et integrale, niſi ſit algebraicum, per logarithmos et angulos exprimetur.

Corollarium 1.

95. Hac ſubſtitutione generalius negotium confici poterit, ſi poſito $(a+bx)^{\frac{1}{\nu}}=u$, littera V denotet functionem quamcunque rationalem binarum quantitatum x et u; cum enim poſito $x=\frac{u^{\nu}-a}{b}$, fiat V functio rationalis ipſius u, formula $V\partial x=\frac{\nu}{b}Vu^{\nu-1}\partial u$, erit rationalis.

Corollarium 2.

96. Quin etiam ſi binae irrationalitates eiusdem quantitatis $a+bx$, ſcilicet $(a+bx)^{\frac{1}{\nu}}=u$ et $(a+bx)^{\frac{1}{n}}=v$, ingrediantur in formulam $X\partial x$, poſito $a+bx=z^{n\nu}$ fit $x=\frac{z^{n\nu}-a}{b}$, $u=z^{n}$, et $v=z^{\nu}$; vnde cum X fiat functio

 ratio-

rationalis ipſius z, et $\partial x = \frac{n\nu}{b} z^{n\nu - 1} \partial z$, hac ſubſtitutione formula $X \partial x$ euadet rationalis.

Corollarium 3.

97. Eodem modo intelligitur, ſi poſito

$$(a + b x)^{\frac{1}{\lambda}} = u,\ (a + b x)^{\frac{1}{\mu}} = v,\ (a + b x)^{\frac{1}{\nu}} = t \text{ etc.}$$

littera X denotet functionem quamcunque rationalem quantitatum x, u, v, t etc. formulam differentialem $X \partial x$, rationalem reddi facto $a + b x = z^{\lambda\mu\nu}$; erit enim

$$x = \frac{z^{\lambda\mu\nu} - a}{b};\ u = z^{\mu\nu};\ v = z^{\lambda\nu};\ t = z^{\lambda\mu} \text{ etc. et}$$

$$\partial x = \frac{\lambda\mu\nu}{b} z^{\lambda\mu\nu - 1} \partial z.$$

Exemplum.

98. Propoſita hac formula $\partial y = \frac{x \partial x}{\sqrt[3]{(1+x)} - \sqrt{(1+x)}}$, facto $1 + x = z^6$, reperitur $\partial y = - \frac{6 z^3 \partial z (1 - z^6)}{1 - z}$, ſeu

$$\partial y = - 6 \partial z (z^3 + z^4 + z^5 + z^6 + z^7 + z^8):$$

hincque integrando

$$y = C - \tfrac{3}{2} z^4 - \tfrac{6}{5} z^5 - z^6 - \tfrac{6}{7} z^7 - \tfrac{3}{4} z^8 - \tfrac{2}{3} z^9,$$

et reſtituendo

$$y = C - \tfrac{3}{2}\sqrt[3]{(1+x)^2} - \tfrac{6}{5}\sqrt[6]{(1+x)^5} - 1 - x - \tfrac{6}{7}(1+x)\sqrt[6]{(1+x)}$$
$$- \tfrac{3}{4}(1+x)\sqrt[3]{(1+x)} - \tfrac{2}{3}(1+x)\sqrt{(1+x)}$$

ita vt integrale adeo algebraice exhibeatur.

Problema 8.

99. Propoſita formula differentiali $X \partial x \left(\frac{a + b x}{f + g x}\right)^{\frac{\mu}{\nu}}$, denotante X functionem rationalem quamcunque ipſius x, eam ab irrationalitate liberare.

Solu-

Solutio.

Posito $\frac{a+bx}{f+gx} = z^\nu$, fit $(\frac{a+bx}{f+gx})^{\frac{\mu}{\nu}} = z^\mu$, et

$$x = \frac{a - fz^\nu}{gz^\nu - b}, \text{ atque } \partial x = \frac{\nu(bf - ag)z^{\nu-1}\partial z}{(gz^\nu - b)^2}:$$

sicque loco X prodibit functio rationalis ipsius z, qua posita $= Z$, erit formula nostra differentialis

$$= \frac{\nu(bf - ag)Zz^{\mu+\nu-1}\partial z}{(gz^\nu - b)^2},$$

quae cum sit rationalis, per praecepta Cap. I. integrari poterit.

Corollarium 1.

100. Posito $(\frac{a+bx}{f+gx})^{\frac{1}{\nu}} = u$, si X fuerit functio quaecunque rationalis binarum quantitatum x et u, formula differentialis $X\partial x$ per substitutionem vsurpatam in rationalem transformabitur, cuius propterea integratio constat.

Corollarium 2.

101. Si X fuerit functio rationalis tam ipsius x, quam quantitatum quotcunque huiusmodi

$$(\frac{a+bx}{f+gx})^{\frac{1}{\lambda}} = u, \; (\frac{a+bx}{f+gx})^{\frac{1}{\mu}} = v, \; (\frac{a+bx}{f+gx})^{\frac{1}{\nu}} = t$$

tum formula differentialis $X\partial x$ rationalis reddetur, adhibita substitutione $\frac{a+bx}{f+gx} = z^{\lambda\mu\nu}$, vnde fit

$$x = \frac{a - fz^{\lambda\mu\nu}}{gz^{\lambda\mu\nu} - b}; \text{ et } u = z^{\mu\nu}; \; v = z^{\lambda\nu}; \; t = z^{\lambda\mu}.$$

Scholion 1.

102. His casibus reductio ad rationalitatem ideo succedit, etiamsi plures formulae irrationales insint, quod eae omnes

nes simul per eandem substitutionem rationales efficiantur, indeque etiam ipsa quantitas x per nouam variabilem z rationaliter exprimetur. Sin autem differentiale propositum duas eiusmodi formulas irrationales contineat, quae non ambae simul ope eiusdem substitutionis rationales reddi queant, etiamsi hoc in vtraque seorsim fieri possit, reductio locum non habet, nisi forte ipsum differentiale in duas partes dispesci liceat, quarum vtraque vnam tantum formulam irrationalem complectatur. Veluti si proposita sit haec formula differentialis

$$\partial y = \frac{\partial x}{\sqrt{(1+xx)}-\sqrt{(1-xx)}}$$

eius numeratorem ac denominatorem per $\sqrt{(1+xx)}+\sqrt{(1-xx)}$ multiplicando, fit

$$\partial y = \frac{\partial x\sqrt{(1+xx)}}{2xx} + \frac{\partial x\sqrt{(1-xx)}}{2xx},$$

cuius vtraque pars seorsim rationalis reddi et integrari potest. Reperitur autem:

$$y = C - \frac{\sqrt{(1-xx)}-\sqrt{(1+xx)}}{2x} + \tfrac{1}{2}l[x+\sqrt{(1+xx)}]$$
$$- \tfrac{1}{2}\,\text{Arc. tang.}\frac{x}{\sqrt{(1-xx)}}.$$

Commodissime autem ibi irrationalitas tollitur, si in parte priori ponatur $\sqrt{(1+xx)} = px$, in posteriori $\sqrt{(1-xx)} = qx$. Etsi enim hinc fit

$$x = \frac{1}{\sqrt{(pp-1)}} \text{ et } x = \frac{1}{\sqrt{(1+qq)}},$$

tamen oritur rationaliter

$$\partial y = \frac{-pp\partial p}{2(pp-1)} - \frac{qq\partial q}{2(1+qq)}.$$

Scholion 2.

103. Circa formulas generales, quae ab irrationalitate liberari queant, vix quicquam amplius praecipere licet: dummodo hunc casum addamus, quo functio X binas huiusmodi formulas radicales $\sqrt{(a+bx)}$ et $\sqrt{(f+gx)}$ complectitur.

Po-

Posito enim $(a+bx)=(f+gx)tt$, fit $x=\frac{a-ftt}{gtt-b}$, atque

$$\sqrt{(a+bx)}=\frac{t\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}};\ \sqrt{(f+gx)}=\frac{\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}};$$

et in formula differentiali vnica tantum formula irrationalis erit $\sqrt{(gtt-b)}$, quae noua substitutione facile tolletur, per ea quae Problemate 6. tradidimus. Vt igitur ad alia pergamus, imprimis considerari meretur haec formula differentialis

$$x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}},$$

cuius ob simplicitatem vsus per vniuersam analysin est amplissimus; vbi quidem sumimus, litteras m, n, μ, ν numeros integros denotare, nisi enim tales essent, facile ad hanc formam reducerentur. Veluti si haberemus $x^{-\frac{1}{3}}\partial x(a+b\sqrt{x})^{\frac{\mu}{\nu}}$, statui oportet $x=u^6$, hinc $\partial x=6u^5\partial u$: vnde prodit

$$6u^3\partial u(a+bu^3)^{\frac{\mu}{\nu}}.$$

Tum vero pro n valorem positiuum assumere licet: si enim esset negatiuus, puta

$$x^{m-1}\partial x(a+bx^{-n})^{\frac{\mu}{\nu}},$$

ponatur $x=\frac{1}{u}$, fietque formula

$$-u^{-m-1}\partial u(a+bu^n)^{\frac{\mu}{\nu}},$$

similis principali; quae ergo quibus casibus ab irrationalitate liberari queat, inuestigemus.

Problema 9.

104. Definire casus, quibus formulam differentialem

$$x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}},$$

ad rationalitatem perducere liceat.

Solutio.

Primo patet, si fuerit $\nu = 1$, seu $\frac{\mu}{\nu}$ numerus integer, formulam per se fore rationalem, neque substitutione opus esse. At si $\frac{\mu}{\nu}$ sit fractio, substitutione est vtendum, eaque duplici.

I. Ponatur $a + b x^n = u^\nu$, vt fiat $(a + b x^n)^{\frac{\mu}{\nu}} = u^\mu$, erit

$$x^n = \frac{u^\nu - a}{b}, \text{ hinc } x^m = \left(\frac{u^\nu - a}{b}\right)^{\frac{m}{n}}, \text{ ideoque}$$

$$x^{m-1} \partial x = \frac{\nu}{nb} u^{\nu - 1} \partial u \left(\frac{u^\nu - a}{b}\right)^{\frac{m-n}{n}}:$$

vnde formula nostra fiet

$$\frac{\nu}{nb} u^{\mu + \nu - 1} \partial u \left(\frac{u^\nu - a}{b}\right)^{\frac{m-n}{n}}.$$

Hinc ergo patet, quoties exponens $\frac{m-n}{n}$ seu $\frac{m}{n}$ fuerit numerus integer siue positiuus, siue negatiuus, hanc formulam esse rationalem.

II. Ponatur $a + b x^n = x^n z^\nu$, vt fiat

$$x^n = \frac{a}{z^\nu - b}, \text{ et } (a + b x^n)^{\frac{\mu}{\nu}} = \frac{a^{\frac{\mu}{\nu}} z^\mu}{(z^\nu - b)^{\frac{\mu}{\nu}}}; \text{ tum}$$

$$x^m = \frac{a^{\frac{m}{n}}}{(z^\nu - b)^{\frac{m}{n}}}, \text{ hinc } x^{m-1} \partial x = \frac{-\nu a^{\frac{m}{n}} z^{\nu - 1} \partial z}{n (z^\nu - b)^{\frac{m}{n} + 1}}.$$

Ideoque formula nostra erit

$$\frac{-\nu a^{\frac{m}{n} + \frac{\mu}{\nu}} z^{\mu + \nu - 1} \partial z}{n (z^\nu - b)^{\frac{m}{n} + \frac{\mu}{\nu} + 1}}.$$

Ex

Ex quo patet hanc formam fore rationalem, quoties $\frac{m}{n} + \frac{\mu}{\nu}$ fuerit numerus integer. Facile autem intelligitur, alias substitutiones huic scopo idoneas excogitari non posse.

Quare concludimus formulam irrationalem hanc

$$x^{m-1}\partial x (a + b x^n)^{\frac{\mu}{\nu}}$$

ab irrationalitate liberari posse, si fuerit vel $\frac{m}{n}$, vel $\frac{m}{n} + \frac{\mu}{\nu}$ numerus integer.

Corollarium 1.

105. Si sit $\frac{m}{n}$ numerus integer, casus per se est facilis; ponatur enim $m = i n$, et sit $x^n = v$, erit $x^m = v^i$; ideoque formula nostra $\frac{i}{m} v^{i-1} \partial v (a + b v)^{\frac{\mu}{\nu}}$, quae per Problema 7. expeditur.

Corollarium 2.

106. At si $\frac{m}{n}$ non est numerus integer, vt reductio ad rationalitatem locum habeat, necesse est vt $\frac{m}{n} + \frac{\mu}{\nu}$ sit numerus integer: quod fieri nequit, nisi sit $\nu = n$, ideoque $m + \mu$ multiplum debet esse ipsius $n = \nu$.

Corollarium 3.

107. Quod si ergo haec formula

$$x^{m-1}\partial x (a + b x^n)^{\frac{\mu}{\nu}},$$

ad rationalitatem reduci queat, etiam haec formula

$$x^{m \pm \alpha n - 1}\partial x (a + b x^n)^{\frac{\mu}{\nu} \pm \beta},$$

eandem reductionem admittet; quicunque numeri integri pro α et β assumantur. Vnde ad casus reducibiles cognoscendos sufficit ponere $m < n$ et $\mu < \nu$.

Corollarium 4.

108. Si $m = 0$, haec formula $\frac{\partial x}{x}(a + b x^n)^{\frac{\mu}{\nu}}$ ſemper per caſum primum ad rationalitatem reducitur, ponendo

$$x^n = \frac{u^\nu - a}{b};$$

transformatur enim in hanc

$$\frac{\nu b u^{\mu + \nu - 1} \partial u}{n(u^\nu - a)}.$$

Scholion 1.

109. Quoniam formula $x^{m-1} \partial x (a + b x^n)^{\frac{\mu}{\nu}}$, quoties eſt $m = i n$, denotante i numerum integrum ſiue poſitiuum ſiue negatiuum quemcunque, ſemper ad rationalitatem reduci poteſt, hicque caſus per ſe ſunt perſpicui, reliquos caſus hanc reductionem admittentes accuratius contemplari operae pretium videtur. Quem in finem ſtatuamus $\nu = n$ et $m < n$, item $\mu < n$, ac neceſſe eſt vt ſit $m + \mu = n$: vnde ſequentes formae in genere ſuo ſimpliciſſimae, quae quidem ad rationalitatem reduci queant, obtinentur.

I. $\partial x (a + b x^2)^{\frac{1}{2}}$;

II. $\partial x (a + b x^3)^{\frac{2}{3}}$; $x \partial x (a + b x^3)^{\frac{1}{3}}$;

III. $\partial x (a + b x^4)^{\frac{3}{4}}$; $x x \partial x (a + b x^4)^{\frac{1}{4}}$;

IV. $\partial x (a + b x^5)^{\frac{4}{5}}$; $x \partial x (a + b x^5)^{\frac{3}{5}}$; $x^2 \partial x (a + b x^5)^{\frac{2}{5}}$;
$x^3 \partial x (a + b x^5)^{\frac{1}{5}}$;

V. $\partial x (a + b x^6)^{\frac{5}{6}}$; $x^4 \partial x (a + b x^6)^{\frac{1}{6}}$;

vnde etiam hae reductionem admittent:

$x \pm$

$x^{\pm 2\alpha}\,\partial x\,(a+b\,x^2)^{\frac{1}{2}\pm\beta}$;

$x^{\pm 3\alpha}\,\partial x\,(a+b\,x^3)^{\frac{2}{3}\pm\beta}$; $x^{1\pm 3\alpha}\,\partial x\,(a+b\,x^3)^{\frac{1}{3}\pm\beta}$;

$x^{\pm 4\alpha}\,\partial x\,(a+b\,x^4)^{\frac{3}{4}\pm\beta}$; $x^{2\pm 4\alpha}\,\partial x\,(a+b\,x^4)^{\frac{1}{4}\pm\beta}$;

$x^{\pm 5\alpha}\,\partial x\,(a+b\,x^5)^{\frac{4}{5}\pm\beta}$; $x^{1\pm 5\alpha}\,\partial x\,(a+b\,x^5)^{\frac{3}{5}\pm\beta}$;

$x^{2\pm 5\alpha}\,\partial x\,(a+b\,x^5)^{\frac{2}{5}\pm\beta}$; $x^{3\pm 5\alpha}\,\partial x\,(a+b\,x^5)^{\frac{1}{5}\pm\beta}$;

$x^{\pm 6\alpha}\,\partial x\,(a+b\,x^6)^{\frac{5}{6}\pm\beta}$; $x^{4\pm 6\alpha}\,\partial x\,(a+b\,x^6)^{\frac{1}{6}\pm\beta}$.

Scholion 2.

110. Verum etiamsi formula $x^{m-1}\,\partial x\,(a+b\,x^n)^{\frac{\mu}{\nu}}$, ab irrationalitate liberari nequeat, tamen semper omnium harum formularum $x^{m\pm n\alpha-1}\,\partial x\,(a+b\,x^n)^{\frac{\mu}{\nu}\pm\beta}$, integrationem ad eam reducere licet, ita vt illius integrali tanquam cognito spectato, etiam harum integralia assignari queant. Quae reductio cum in Analysi summam afferat vtilitatem, eam hic exponere necesse erit. Caeterum hic affirmare haud dubitamus, praeter eos casus, quos reductionem ad rationalitatem admittere hic ostendimus, nullos alios existere, qui vlla substitutione adhibita ab irrationalitate liberari queant. Proposita enim hac formula $\frac{\partial x}{\sqrt{(a+b\,x^3)}}$, nulla functio rationalis ipsius z loco x poni potest, vt $a+b\,x^3$ extractionem radicis quadratae admittat: obiici quidem potest, scopo satisfieri posse, etiamsi loco x functio irrationalis ipsius z substituatur, dummodo similis irrationalitas in denominatore $\sqrt{(a+b\,x^3)}$ contineatur, qua illa numeratorem ∂x afficiens destruatur: quemadmodum fit in hac formula $\frac{\partial x}{\sqrt[3]{(a+b\,x^3)}}$, adhibendo substitutionem

$$x=\frac{\sqrt[3]{a}}{\sqrt[3]{(z^3+b)}},$$

 ve-

verum quod hic commode vſu venit, nullo modo perſpicitur, quomodo idem illo caſu euenire poſſit. Hoc tamen minime pro demonſtratione haberi volo.

Problema 10.

111. Integrationem formulae

$$\int x^{m+n-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}},$$

perducere ad integrationem huius formulae

$$\int x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}}.$$

Solutio.

Conſideretur functio $x^m(a+bx^n)^{\frac{\mu}{\nu}+1}$, cuius differentiale cum ſit

$$\left(max^{m-1}\partial x+mbx^{m+n-1}\partial x+\tfrac{n(\mu+\nu)}{\nu}bx^{m+n-1}\partial x\right)(a+bx^n)^{\frac{\mu}{\nu}},$$

erit

$$x^m(a+bx^n)^{\frac{\mu}{\nu}+1}=ma\int x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}}$$
$$+\tfrac{(m\nu+n\mu+n\nu)b}{\nu}\int x^{m+n-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}}:$$

vnde elicitur

$$\int x^{m+n-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}}=\frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{(m\nu+n\mu+n\nu)b}$$
$$-\frac{m\nu a}{(m\nu+n\mu+n\nu)b}\int x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 1.

112. Cum inde quoque ſit

$$\int x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}}=\frac{x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{ma}$$

— (m

$$-\frac{(m\nu+n\mu+n\nu)b}{m\nu a}\int x^{m+n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}},$$

loco m scribamus $m-n$, et habebimus hanc reductionem:

$$\int x^{m-n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}=\frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{(m-n)a}$$
$$-\frac{(m\nu+n\mu)b}{(m-n)\nu a}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 2.

113. Concesso ergo integrali $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, etiam harum formularum $\int x^{m\pm n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, similique modo vlterius progrediendo omnium harum formularum

$$\int x^{m\pm in+1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

integralia exhiberi possunt.

Problema 11.

114. Integrationem formulae $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1}$ ad integrationem huius $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$ perducere.

Solutio.

Functionis $x^m(a+bx^n)^{\frac{\mu}{\nu}+1}$ differentiale hoc modo exhiberi potest

$$\left(ma-\frac{(m\nu+n\mu+n\nu)a}{\nu}\right)x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$
$$+\frac{m\nu+n\mu+n\nu}{\nu}x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1},$$

vnde concluditur

x^m

$$x^m (a+bx^n)^{\frac{\mu}{\nu}+1} = -\frac{(n\mu+n\nu)a}{\nu}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$
$$+\frac{m\nu+n\mu+n\nu}{\nu}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1},$$

quocirca habebimus:

$$\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{\nu x^m (a+bx^n)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)}$$
$$+\frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 1.

115. Deinde ex eadem aequatione elicimus:

$$\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}} = \frac{-\nu x^m (a+bx^n)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)a}$$
$$+\frac{m\nu+n(\mu+\nu)}{n(\mu+\nu)a}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1}.$$

Scribamus iam $\mu-\nu$ loco μ, vt nancifcamur hanc reductionem

$$\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{-\nu x^m (a+bx^n)^{\frac{\mu}{\nu}}}{n\mu a}$$
$$+\frac{m\nu+n\mu}{n\mu a}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 2.

116. Conceffo ergo integrali $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, etiam harum formularum $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}\pm 1}$, et vlterius progrediendo, harum $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$ integralia exhiberi poffunt, denotante β numerum integrum quemcunque.

Corol-

Corollarium 3.

117. His cum praecedentibus coniunctis, ad integrationem $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, omnia haec integralia

$$\int x^{m\pm\alpha n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$$

reuocari possunt, quae ergo omnia ab eadem functione transcendente pendent.

Scholion 1.

118. Ex formae $x^m(a+bx^n)^{\frac{\mu}{\nu}}$ differentiale ita disposito

$$mx^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}+\frac{n\mu}{\nu}bx^{m+n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1}$$

deducimus hanc reductionem:

$$\int x^{m+n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1}=\frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b}$$
$$-\frac{m\nu}{n\mu b}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}:$$

ac praeterea hanc inuersam, pro m et μ scribendo $m-n$ et $\mu+\nu$:

$$\int x^{m-n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1}=\frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{m-n}$$
$$-\frac{n(\mu+\nu)b}{\nu(m-n)}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}.$$

Hinc scilicet vna operatione absoluitur reductio, cum superiores formulae duplicem reductionem exigant; ex quo sex reductiones sumus nacti, omnino memorabiles, quas idcirco coniunctim conspectui exponamus.

I I. $\int$

$$\text{I. } \int x^{m+n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}} = \frac{\nu x^m (a+bx^n)^{\frac{\mu}{\nu}+1}}{[m\nu+n(\mu+\nu)]\,b} - \frac{m\nu a}{[m\nu+n(\mu+\nu)]\,b}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{II. } \int x^{m-n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{(m-n)\,a} - \frac{(m\nu+n\mu)\,b}{(m-n)\,\nu a}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{III. } \int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{-\nu x^m (a+bx^n)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)} + \frac{n(\mu+\nu)\,a}{m\nu+n(\mu+\nu)}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{IV. } \int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{-\nu x^m (a+bx^n)^{\frac{\mu}{\nu}}}{n\mu a} + \frac{m\nu+n\mu}{n\mu a}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{V. } \int x^{m+n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu x^m (a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b} - \frac{m\nu}{n\mu b}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{VI. } \int x^{m-n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{m-n} - \frac{n(\mu+\nu)\,b}{\nu(m-n)}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}.$$

Scho-

Scholion 2.

119. Circa has reductiones primo obseruandum est, formulam priorem algebraice esse integrabilem, si coefficiens posterioris euanescat. Ita fit

pro I. si $m=0 \ldots \int x^{n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}} = \frac{\nu(a+bx^n)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)b}$

pro II. si $\frac{\mu}{\nu}=\frac{-m}{n} \ldots \int x^{m-n-1}\partial x\,(a+bx^n)^{\frac{-m}{n}} = \frac{x^{m-n}(a+bx^n)^{\frac{-m}{n}+1}}{(m-n)a}$

pro IV. si $\frac{\mu}{\nu}=\frac{-m}{n} \ldots \int x^{m-1}\partial x\,(a+bx^n)^{\frac{-m}{n}-1} = \frac{x^m(a+bx^n)^{\frac{-m}{n}}}{ma}$

pro V. si $m=0 \ldots \int x^{n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu(a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b}$.

Deinde etiam casus notari merentur, quibus coefficiens postremae formulae fit infinitus; tum enim reductio cessat, et prior formula peculiare habet integrale seorsim euoluendum.

In prima hoc euenit si $\frac{\mu+\nu}{\nu}=\frac{-m}{n}$, et formula

$$\int x^{m+n-1}\partial x\,(a+bx^n)^{\frac{-m}{n}-1},$$

posito $a+bx^n=x^n z^n$, seu $x^n=\frac{a}{z^n-b}$, abit in $-\frac{z^{-m-1}\partial z}{z^n-b}$, cuius integrale per caput primum definiri debet.

In secunda euenit si $m=n$, et formula $\int\frac{\partial x}{x}(a+bx^n)^{\frac{\mu}{\nu}}$, posito $a+bx^n=z^\nu$, seu $x^n=\frac{z^\nu-a}{b}$, abit in $\frac{\nu z^{\mu+\nu-1}\partial z}{n(z^\nu-a)}$;

In tertia euenit, si $\frac{\mu}{\nu} = \frac{-m}{n} - 1$, et formula

$$\int x^{m-1} \partial x (a + b x^n)^{\frac{-m}{n}},$$

posito $a + b x^n = x^n z^n$, seu $x^n = \frac{a}{z^n - b}$, abit in $\int \frac{-z^{-m-n-1} \partial z}{z^n - b}$, seu posito $z = \frac{1}{u}$, in

$$\int \frac{u^{m+2n-1} \partial u}{1 - b u^n} = \frac{-u^{m+n}}{(m+n) b} - \frac{u^m}{m b b} + \frac{1}{b b} \int \frac{u^{m-1} \partial u}{a - b u^n}.$$

In quarta euenit, si $\mu = 0$, et formula $\int \frac{x^{m-1} \partial x}{a + b x^n}$ per se est rationalis.

In quinta idem euenit, si $\mu = 0$.

In sexta autem, si $m = n$, et formula $\int \frac{\partial x}{x} (a + b x^n)^{\frac{\mu}{\nu} + 1}$, posito $a + b x^n = z^\nu$, abit in $\frac{\nu}{n} \int \frac{z^{\mu + 2\nu - 1} \partial z}{z^\nu - a}$.

Exemplum 1.

120. *Inuenire integrale huius formulae* $\int \frac{x^{m-1} \partial x}{\sqrt{(1 - x x)}}$, *pro numeris positiuis exponenti* m *datis.*

Hic ob ob $a = 1$, $b = -1$, $n = 2$, $\mu = -1$, $\nu = 2$, prima reductio dat:

$$\int \frac{x^{m+1} \partial x}{\sqrt{(1 - x x)}} = \frac{-x^m \sqrt{(1 - x x)}}{m + 1} + \frac{m}{m+1} \int \frac{x^{m-1} \partial x}{\sqrt{(1 - x x)}}:$$

hinc prout pro m sumantur numeri vel impares vel pares, obtinebimus.

Pro

Pro numeris imparibus:

$$\int \frac{xx\,\partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{2}x\sqrt{(1-xx)} + \tfrac{1}{2}\int \frac{\partial x}{\sqrt{(1-xx)}}$$
$$\int \frac{x^4\,\partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{4}x^3\sqrt{(1-xx)} + \tfrac{3}{4}\int \frac{x^2\,\partial x}{\sqrt{(1-xx)}}$$
$$\int \frac{x^6\,\partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{6}x^5\sqrt{(1-xx)} + \tfrac{5}{6}\int \frac{x^4\,\partial x}{\sqrt{(1-xx)}}$$

Pro numeris paribus

$$\int \frac{x^3\,\partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{3}x^2\sqrt{(1-xx)} + \tfrac{2}{3}\int \frac{x\,\partial x}{\sqrt{(1-xx)}}$$
$$\int \frac{x^5\,\partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{5}x^4\sqrt{(1-xx)} + \tfrac{4}{5}\int \frac{x^3\,\partial x}{\sqrt{(1-xx)}}$$
$$\int \frac{x^7\,\partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{7}x^6\sqrt{(1-xx)} + \tfrac{6}{7}\int \frac{x^5\,\partial x}{\sqrt{(1-xx)}}$$

etc.

Cum nunc sit $\int \frac{\partial x}{\sqrt{(1-xx)}} = \text{Arc. sin. } x$, et

$$\int \frac{x\,\partial x}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)},$$

habebimus sequentia integralia:

Pro ordine priore:

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \text{Arc. sin. } x$$
$$\int \frac{xx\,\partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{2}x\sqrt{(1-xx)} + \tfrac{1}{2}\,\text{Arc. sin. } x$$
$$\int \frac{x^4\,\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{4}x^3 + \tfrac{1.3}{2.4}x\right)\sqrt{(1-xx)} + \tfrac{1.3}{2.4}\,\text{Arc. sin. } x$$
$$\int \frac{x^6\,\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{6}x^5 + \tfrac{1.5}{4.6} + x^3\tfrac{1.3.5}{2.4.6}x\right)\sqrt{(1-xx)} + \tfrac{1.3.5}{2.4.6}\,\text{Arc. sin. } x$$
$$\int \frac{x^8\,\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{8}x^7 + \tfrac{1.7}{6.8}x^5 + \tfrac{1.5.7}{4.6.8}x^3 + \tfrac{1.3.5.7}{2.4.6.8}x\right)\sqrt{(1-xx)} + \tfrac{1.3.5.7}{2.4.6.8}\,\text{Arc. sin. } x.$$

Pro ordine posteriore:

$$\int \frac{x\,\partial x}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)}$$
$$\int \frac{x^3\,\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{3}x^2 + \tfrac{2}{3}\right)\sqrt{(1-xx)}$$
$$\int \frac{x^5\,\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{5}x^4 + \tfrac{1.4}{3.5}x^2 + \tfrac{2.4}{3.5}\right)\sqrt{(1-xx)}$$
$$\int \frac{x^7\,\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{7}x^6 + \tfrac{1.6}{5.7}x^4 + \tfrac{1.4.6}{3.5.7}x^2 + \tfrac{2.4.6}{3.5.7}\right)\sqrt{(1-xx)}.$$

Corollarium 1.

121. In genere ergo formula $\int \frac{x^{2i}\partial x}{\sqrt{(1-xx)}}$, si ponamus breuitatis gratia $\frac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i} = J$, habebimus hoc integrale:

$$\int \frac{x^{2i}\partial x}{\sqrt{(1-xx)}} = J\ \text{Arc. sin.}\ x$$
$$-J(x+\tfrac{2}{3}x^3+\tfrac{2.4}{3.5}x^5+\tfrac{2.4.6}{3.5.7}x^7\ldots+\tfrac{2.4.6\ldots(2i-2)}{3.5.7\ldots(2i-1)}x^{2i-1})\sqrt{(1-xx)}.$$

Corollarium 2.

222. Simili modo pro formula $\int \frac{x^{2i+1}\partial x}{\sqrt{(1-xx)}}$, si ponamus breuitatis ergo $\frac{2.4.6\ldots 2i}{3.5.7\ldots(2i+1)} = K$, habebimus hoc integrale:

$$\int \frac{x^{2i+1}\partial x}{\sqrt{(1-xx)}} = K -$$
$$K(1+\tfrac{1}{2}x^2+\tfrac{1.3}{2.4}x^4+\tfrac{1.3.5}{2.4.6}x^6+\ldots\tfrac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i}x^{2i})\sqrt{(1-xx)}$$

vt integrale euanescat posito $x=0$.

Exemplum 2.

123. *Inuenire integrale formulae* $\int \frac{x^{m-1}\partial x}{\sqrt{(1-xx)}}$, *casibus quibus pro* m *numeri negatiui assumuntur.*

Hic vtendum est secunda reductione quae dat:

$$\int \frac{x^{m-3}\partial x}{\sqrt{(1-xx)}} = \frac{x^{m-2}\sqrt{(1-xx)}}{m-2} + \frac{m-1}{m-2}\int \frac{x^{m-1}\partial x}{\sqrt{(1-xx)}}:$$

vnde patet si $m=1$, fore $\int \frac{\partial x}{xx\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{x}$. Deinde

si

si $m = 2$, formula $\int \frac{\partial x}{x\sqrt{(1-xx)}}$, facta substitutione $1 - xx = zz$, abit in $\frac{-\partial z}{1-zz}$ cuius integrale est

$$-\tfrac{1}{2} l \frac{1+z}{1-z} = -\tfrac{1}{2} l \frac{1+\sqrt{(1-xx)}}{1-\sqrt{(1-xx)}} = -l \frac{1+\sqrt{(1-xx)}}{x}:$$

vnde duplicem seriem integrationum elicimus:

$$\int \frac{\partial x}{x\sqrt{(1-xx)}} = -l \frac{1+\sqrt{(1-xx)}}{x} = l \frac{1-\sqrt{(1-xx)}}{x};$$

$$\int \frac{\partial x}{x^3\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{2xx} + \tfrac{1}{2}\int \frac{\partial x}{x\sqrt{(1-xx)}};$$

$$\int \frac{\partial x}{x^5\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{4x^4} + \tfrac{3}{4}\int \frac{\partial x}{x^3\sqrt{(1-xx)}};$$

$$\int \frac{\partial x}{x^7\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{6x^6} + \tfrac{5}{6}\int \frac{\partial x}{x^5\sqrt{(1-xx)}}.$$

etc.

$$\int \frac{\partial x}{xx\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{x};$$

$$\int \frac{\partial x}{x^4\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{3x^3} + \tfrac{2}{3}\int \frac{\partial x}{xx\sqrt{(1-xx)}};$$

$$\int \frac{\partial x}{x^6\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{5x^5} + \tfrac{4}{5}\int \frac{\partial x}{x^4\sqrt{(1-xx)}}.$$

etc.

Hinc erit, vt in binis praecedentibus corollariis

$$\int \frac{\partial x}{x^{2i+1}\sqrt{(1-xx)}} = lJ \frac{1-\sqrt{(1-xx)}}{x} - J\left[\frac{1}{xx} + \frac{2}{3x^4} + \frac{2.4}{3.5x^6} + \ldots \right.$$
$$\left. \ldots\ldots + \frac{2.4\ldots.(2i-2)}{3.5\ldots.(2i-1)x^{2i}}\right]\sqrt{(1-xx)};$$

$$\int \frac{\partial x}{x^{2i}\sqrt{(1-xx)}} = C - K\left[\frac{1}{x} + \frac{1}{2x^3} + \frac{1.3}{2.4x^5} + \ldots\ldots \right.$$
$$\left. \ldots\ldots + \frac{1.3\ldots.(2i-1)}{2.4\ldots.2ix^{2i+1}}\right]\sqrt{(1-xx)}.$$

Scholion 1.

124. Hinc iam facile integralia formularum

$$\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}}$$

tam

tam pro omnibus numeris m, quam pro imparibus μ assignari poterunt. Reductiones autem nostrae generales ad hunc casum accommodatae sunt:

$$\text{I. } \int x^{m+1}\partial x(1-xx)^{\frac{\mu}{2}} = \frac{-x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{m}{m+\mu+2}\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}};$$

$$\text{II. } \int x^{m-3}\partial x(1-xx)^{\frac{\mu}{2}} = \frac{x^{m-2}(1-xx)^{\frac{\mu}{2}+1}}{m-2} + \frac{m+\mu}{m-2}\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}};$$

$$\text{III. } \int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}+1} = \frac{x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{\mu+2}{m+\mu+2}\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}};$$

$$\text{IV. } \int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m(1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m+\mu}{\mu}\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}};$$

$$\text{V. } \int x^{m+1}\partial x(1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m(1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m}{\mu}\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}};$$

$$\text{VI. } \int x^{m-3}\partial x(1-xx)^{\frac{\mu}{2}+1} = \frac{x^{m-2}(1-xx)^{\frac{\mu}{2}+1}}{m-2} + \frac{\mu+2}{m-2}\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}}.$$

Posito

Posito enim $\mu = -1$, quatuor posteriores dant:

$$\int x^{m-1}\partial x\sqrt{(1-xx)} = \frac{x^m\sqrt{(1-xx)}}{m+1} + \frac{1}{m+1}\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

$$\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)^3}} = \frac{x^m}{\sqrt{(1-xx)}} - (m-1)\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

$$\int\frac{x^{m+1}\partial x}{\sqrt{(1-xx)^3}} = \frac{x^m}{\sqrt{(1-xx)}} - m\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

$$\int x^{m-3}\partial x\sqrt{(1-xx)} = \frac{x^{m-2}\sqrt{(1-xx)}}{m-x} + \frac{1}{m-2}\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

vnde integrationes pro casibus $\mu = 1$ et $\mu = -3$ eliciuntur, indeque porro reliqui.

Scholion 2.

125. Pro aliis formulis irrationalibus magis complicatis vix regulas dare licet, quibus ad formam simpliciorem reduci queant: et quoties eiusmodi formulae occurrant, reductio, si quam admittunt, plerumque sponte se offert. Veluti si formula fuerit huiusmodi $\int\frac{P\,\partial x}{Q^{n+1}}$, siue n sit numerus integer siue fractus, semper ad aliam huius formae $\int\frac{S\,\partial x}{Q^n}$, quae vtique simplicior aestimatur reduci potest. Cum enim sit

$$\partial\frac{R}{Q^n} = \frac{Q\,\partial R - nR\,\partial Q}{Q^{n+1}},\ \text{posito}\ \int\frac{P\,\partial x}{Q^{n+1}} = y,\ \text{erit}$$

$$y + \frac{R}{Q^n} = \int\frac{P\,\partial x + Q\,\partial R - nR\,\partial Q}{Q^{n+1}}.$$

Iam definiatur R ita, vt $P\,\partial x + Q\,\partial R - nR\,\partial Q$ per Q fiat diuisibile, vel quia $Q\,\partial R$ iam factorem habet Q, vt fiat $P\,\partial x - nR\,\partial Q = QT\,\partial x$, prodibitque

$$y + \frac{R}{Q^n} = \int \frac{\partial R + T\,\partial x}{Q^n}, \text{ seu}$$

$$\int \frac{P\,\partial x}{Q^{n+1}} = -\frac{R}{Q^n} + \int \frac{\partial R + T\,\partial x}{Q^n}.$$

At ſemper functionem R ita definire licet, vt $P\,\partial x - n\,R\,\partial Q$ factorem Q obtineat, quod etſi in genere praeſtari nequit, tamen rem in exemplis tentando, mox perſpicietur negotium ſemper ſuccedere. Aſſumo autem hic, P et Q eſſe functiones integras, ac talis quoque ſemper pro R erui poterit. Si forte eueniat, vt $\partial R + T\,\partial x = 0$, formula propoſita algebraicum habebit integrale, quod hoc modo reperietur; contra autem haec forma vlterius reduci poterit in alias, vbi denominatoris exponens continuo vnitate diminuatur; ac ſi n ſit numerus integer, negotium tandem reducitur ad huiusmodi formam $\frac{V\,\partial x}{Q}$, quae ſine dubio eſt ſimpliciſſima. Quamobrem cum in hoc capite vix quicquam amplius proferri poſſit, ad integrationem formularum irrationalium iuuandam, methodum easdem integrationes per ſeries infinitas perficiendi exponamus.

ADDITAMENTVM.

Problema.

Propoſita formula $\partial y = [x + \sqrt{(1 + xx)}]^n\,\partial x$, inuenire eius integrale.

Solutio.

Poſito $x + \sqrt{(1 + xx)} = u$, fit $x = \frac{uu - 1}{2u}$, et $\partial x = \frac{\partial u (uu + 1)}{2uu}$: vnde formula noſtra

$$\partial y = \tfrac{1}{2} u^{n-2}\,\partial u (uu + 1),$$

ideo-

ideoque eius integrale

$$y = \frac{u^{n+1}}{2(n+1)} + \frac{u^{n-1}}{2(n-1)} + \text{Const.}$$

quod ergo semper est algebraicum nisi sit vel $n = 1$, vel $n = -1$.

Corollarium 1.

Patet etiam hanc formam latius patentem

$$\partial y = [x + \sqrt{(1 + xx)}]^n X \partial x$$

hoc modo integrari posse, dummodo X fuerit functio rationalis ipsius x. Posito enim $x = \frac{uu-1}{2u}$, pro X prodit functio rationalis ipsius u, quae sit $= U$, hincque fit

$$\partial y = \tfrac{1}{2} U u^{n-2} \partial u (uu + 1),$$

quae formula vel est rationalis, si n sit numerus integer, vel ad rationalitatem facile reducitur, si n sit numerus fractus.

Corollarium 2.

Cum sit $\sqrt{(1 + xx)} = \frac{uu+1}{2u}$; posito $\sqrt{(1+xx)} = v$, etiam haec formula

$$\partial y = [x + \sqrt{(1 + xx)}]^n X \partial x$$

integrabitur, si X fuerit functio rationalis quaecunque quantitatum x et v. Facto enim $x = \frac{uu-1}{2u}$, functio X abit in functionem rationalem ipsius u, qua posita $= U$, habebitur vt ante $\partial y = \tfrac{1}{2} U u^{n-2} \partial u (uu + 1)$.

Exemplum.

Proposita sit formula

$$\partial y = [ax + b\sqrt{(1 + xx)}][x + \sqrt{(1 + xx)}]^n \partial x.$$

Posito $x = \frac{uu - 1}{2u}$, fit

$$\partial y = \left(\frac{a(uu-1) + b(uu+1)}{2u}\right) \times \tfrac{1}{2} u^{n-2} \partial u (uu + 1):$$

seu

$$\partial y = \tfrac{1}{4} u^{n-3} \partial u \left[a(u^4 - 1) + b(u^4 + 2uu + 1)\right],$$

cuius integrale est:

$$y = \frac{a+b}{4(n+2)} u^{n+2} + \frac{b}{2n} u^n + \frac{b-a}{4(n-2)} u^{n-2} + \text{Const.}$$

quae est algebraica, nisi sit vel $n = 2$, vel $n = -2$, vel etiam $n = 0$.

CAPVT III.

DE INTEGRATIONE FORMVLARVM DIFFERENTIALIVM PER SERIES INFINITAS.

Problema 12.

126.

Si X fuerit functio rationalis fracta ipfius x, formulae differentialis $\partial y = X \partial x$ integrale per feriem infinitam exhibere.

Solutio.

Cum X fit functio rationalis fracta, eius valor femper ita euolui poteft, vt fiat

$$X = A x^m + B x^{m+n} + C x^{m+2n} + D x^{m+3n} + E x^{m+4n} + \text{etc.}$$

vbi coefficientes A, B, C, etc. feriem recurrentem conftituent, ex denominatore fractionis determinandam. Multiplicentur ergo finguli termini per ∂x, et integrentur, quo facto integrale y per fequentem feriem exprimetur

$$y = \frac{A x^{m+1}}{m+1} + \frac{B x^{m+n+1}}{m+n+1} + \frac{C x^{m+2n+1}}{m+2n+1} + \text{etc.} + \text{Conft.}$$

vbi fi in ferie pro X occurrat huiusmodi terminus $\frac{M}{x}$, inde in integrale ingredietur terminus $M\, l\, x$.

Scholion.

127. Cum integrale $\int X \partial x$, nifi fit algebraicum, per logarithmos et angulos exprimatur, hinc valores logarithmorum et angulorum per feries infinitas exhiberi poffunt. Cuiusmodi feries cum iam in Introductione plures fint traditae,

 non

non ſolum eaedem, ſed etiam infinitae aliae hic per integrationem erui poſſunt. Hoc exemplis declaraſſe iuuabit, vbi potiſſimum eiusmodi formulas euoluemus, in quibus denominator eſt binomium; tum vero etiam caſus aliquot denominatore trinomino vel multinomio praeditos contemplabimur. Imprimis autem eiusmodi eligemus, quibus fractio in aliam, cuius denominator eſt binomius, transmutari poteſt.

Exemplum 1.

128. *Formulam differentialem* $\frac{\partial x}{a+x}$ *per ſeriem integrare.*

Sit $y = \int \frac{\partial x}{a+x}$, erit $y = l(a+x) +$ Conſt. vnde integrali ita determinato, vt euaneſcat poſito $x = 0$, erit $y = l(a+x) - la$. Iam cum ſit

$$\frac{1}{a+x} = \frac{1}{a} - \frac{x}{a^2} + \frac{xx}{a^3} - \frac{x^3}{a^4} + \frac{x^4}{a^5} - \text{etc.}$$

erit eadem lege integrale definiendo:

$$y = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \text{etc.}$$

vnde colligimus, vti quidem iam conſtat:

$$l(a+x) = la + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \text{etc.}$$

Corollarium 1.

129. Si capiamus x negatiuum, vt ſit $\partial y = \frac{-\partial x}{a-x}$, eodem modo patebit eſſe:

$$l(a-x) = la - \frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \frac{x^4}{4a^4} - \text{etc.}$$

hisque combinandis:

$$l(aa-xx) = 2la - \frac{xx}{aa} - \frac{x^4}{2a^4} - \frac{x^6}{3a^6} - \frac{x^8}{4a^8} - \text{etc. et}$$

$$l\frac{a+x}{a-x} = \frac{2x}{a} + \frac{2x^3}{3a^3} + \frac{2x^5}{5a^5} + \frac{2x^7}{7a^7} + \text{etc.}$$

Corollarium 2.

130. Hae poſteriores ſeries eruuntur per integrationem formularum:

$$\frac{-2x\partial x}{aa-xx} = -2x\partial x\left(\frac{1}{aa}+\frac{xx}{a^4}+\frac{x^4}{a^6}+\text{etc.}\right) \text{ et}$$
$$\frac{2a\partial x}{aa-xx} = 2a\partial x\left(\frac{1}{aa}+\frac{xx}{a^4}+\frac{x^4}{a^6}+\text{etc.}\right).$$

Eſt autem $\int\frac{-2x\partial x}{aa-xx} = l(aa-xx)-l\,aa$, et $\int\frac{2a\partial x}{aa-xx} = l\frac{a+x}{a-x}$, ita vt iam his formulis per ſeries integrandis ſuperſedere poſſimus.

Exemplum 2.

131. *Formulam differentialem $\frac{a\partial x}{aa+xx}$ per ſeriem integrare.*

Sit $\partial y = \frac{a\partial x}{aa+xx}$, et cum ſit $y = \text{Arc. tang.}\frac{x}{a}$, idem angulus ſerie infinita exprimetur. Quia enim habemus:

$$\frac{a}{aa+xx} = \frac{1}{a}-\frac{xx}{a^3}+\frac{x^4}{a^5}-\frac{x^6}{a^7}+\frac{x^8}{a^9}-\text{etc.}$$

erit integrando:

$$y = \text{Arc. tang.}\frac{x}{a} = \frac{x}{a}-\frac{x^3}{3a^3}+\frac{x^5}{5a^5}-\frac{x^7}{7a^7}+\text{etc.}$$

Exemplum 3.

132. *Integralia harum formularum $\frac{\partial x}{1+x^3}$ et $\frac{x\partial x}{1+x^3}$ per ſeries exprimere.*

Cum ſit $\frac{1}{1+x^3} = 1-x^3+x^6-x^9+x^{12}-$ etc. erit

$$\int\frac{\partial x}{1+x^3} = x-\tfrac{1}{4}x^4+\tfrac{1}{7}x^7-\tfrac{1}{10}x^{10}+\tfrac{1}{13}x^{13}-\text{etc. et}$$
$$\int\frac{x\partial x}{1+x^3} = \tfrac{1}{2}x^2-\tfrac{1}{5}x^5+\tfrac{1}{8}x^8-\tfrac{1}{11}x^{11}+\tfrac{1}{14}x^{14}-\text{etc.}$$

Verum per §. 77. habemus per logarithmos et angulos:

$$\int\frac{\partial x}{1+x^3} = \tfrac{1}{3}l(1+x)-\tfrac{2}{3}\text{coſ.}\tfrac{\pi}{3}\,l\sqrt{(1-2x\,\text{coſ.}\tfrac{\pi}{3}+xx)}$$
$$+\tfrac{2}{3}\text{ſin.}\tfrac{\pi}{3}\,\text{Arc. tang.}\frac{x\,\text{ſin.}\frac{\pi}{3}}{1-x\,\text{coſ.}\frac{\pi}{3}}$$
$$\int\frac{x\partial x}{1+x^3} = -\tfrac{1}{3}l(1+x)-\tfrac{2}{3}\text{coſ.}\tfrac{2\pi}{3}\,l\sqrt{(1-2x\,\text{coſ.}\tfrac{\pi}{3}+xx)}$$
$$+\tfrac{2}{3}\text{ſin.}\tfrac{2\pi}{3}\,\text{Arc. tang.}\frac{x\,\text{ſin.}\frac{\pi}{3}}{1-x\,\text{coſ.}\frac{\pi}{3}}.$$

At

At est $\cos.\frac{\pi}{3} = \frac{1}{2}$; $\cos.\frac{2\pi}{3} = -\frac{1}{2}$; $\sin.\frac{\pi}{3} = \frac{\sqrt{3}}{2}$; $\sin.\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$; vnde fit

$$\int \frac{\partial x}{1+x^3} = \tfrac{1}{3}\, l\,(1+x) - \tfrac{1}{3}\, l\, \sqrt{(1-x+xx)} + \tfrac{1}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x}$$
$$\int \frac{x\,\partial x}{1+x^3} = -\tfrac{1}{3}\, l\,(1+x) + \tfrac{1}{3}\, l\, \sqrt{(1-x+xx)} + \tfrac{1}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x}$$

integralibus vt seriebus ita sumtis, vt euanescant posito $x = 0$.

Corollarium 1.

133. His igitur seriebus additis, prodit

$$\tfrac{2}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x} = x + \tfrac{1}{2}xx - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \tfrac{1}{8}x^8 - \tfrac{1}{10}x^{10} - \tfrac{1}{11}x^{11} + \text{etc.}$$

subtracta autem posteriori a priori, fit

$$\tfrac{2}{3}\, l\, \frac{1+x}{\sqrt{(1-x+xx)}} = x - \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{8}x^8 - \tfrac{1}{10}x^{10} + \tfrac{1}{11}x^{11} + \text{etc.}$$

cuius valor etiam est

$$\tfrac{1}{3}\, l\, \frac{(1+x)^2}{1-x+xx} = \tfrac{1}{3}\, l\, \frac{(1+x)^3}{1+x^3}.$$

Corollarium 2.

134. Cum sit $\int \frac{xx\,\partial x}{1+x^3} = \frac{1}{3}\, l\,(1+x^3)$, erit eodem modo

$$\tfrac{1}{3}\, l\,(1+x^3) = \tfrac{1}{3}x^3 - \tfrac{1}{6}x^6 + \tfrac{1}{9}x^9 - \tfrac{1}{12}x^{12} + \text{etc.}$$

qua serie illis adiecta, omnes potestates ipsius x occurrent.

Exemplum 4.

135. *Integrale hoc* $y = \int \frac{(1+xx)\,\partial x}{1+x^4}$ *per seriem exprimere.*

Cum sit $\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + x^{16} - \text{etc.}$ erit

$$y = x + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 + \tfrac{1}{11}x^{11} - \tfrac{1}{13}x^{13} - \tfrac{1}{15}x^{15} + \text{etc.}$$

Verum per §. 82. vbi $m = 1$ et $n = 4$, posito $\frac{\pi}{4} = \omega$, fit integrale idem:

$y =$

$$y = \text{fin.}\,\omega \text{ Arc. tang.} \frac{x \,\text{fin.}\,\omega}{1 - x \cos.\,\omega}$$
$$+ \text{fin.}\,3\,\omega \text{ Arc. tang.} \frac{x \,\text{fin.}\,3\,\omega}{1 - x \cos.\,3\,\omega}.$$

At ob $\frac{\pi}{4} = \omega = 45^\circ$, eft $\text{fin.}\,\omega = \frac{1}{\sqrt{2}}$; $\text{cof.}\,\omega = \frac{1}{\sqrt{2}}$; $\text{fin.}\,3\,\omega = \frac{1}{\sqrt{2}}$; $\text{cof.}\,3\,\omega = \frac{-1}{\sqrt{2}}$; hinc habebimus:

$$y = \frac{1}{\sqrt{2}} \text{ Arc. tang.} \frac{x}{\sqrt{2} - x} + \frac{1}{\sqrt{2}} \text{ Arc. tang.} \frac{x}{\sqrt{2} + x}$$
$$= \frac{1}{\sqrt{2}} \text{ Arc. tang.} \frac{x\sqrt{2}}{1 - xx}.$$

Exemplum 5.

136. *Integrale hoc $y = \int \frac{(1 + x^4)\,\partial x}{1 + x^6}$ per feriem exprimere.*

Cum fit $\frac{1}{1 + x^6} = 1 - x^6 + x^{12} - x^{18} + x^{24} - \text{etc.}$ erit

$$y = x + \tfrac{1}{5} x^5 - \tfrac{1}{7} x^7 - \tfrac{1}{11} x^{11} + \tfrac{1}{13} x^{13} + \tfrac{1}{17} x^{17} - \text{etc.}$$

At per §. 82. vbi $m = 1$, $n = 6$, et $\omega = \frac{\pi}{6} = 30^\circ$, eft

$$y = \tfrac{2}{3} \text{fin.}\,\omega \text{ Arc. tang.} \frac{x\,\text{fin.}\,\omega}{1 - x \cos.\,\omega} + \tfrac{2}{3} \text{fin.}\,3\,\omega \text{ Arc. tang.} \frac{x\,\text{fin.}\,3\,\omega}{1 - x \cos.\,3\,\omega}$$
$$+ \tfrac{2}{3} \text{fin.}\,5\,\omega \text{ Arc. tang.} \frac{x\,\text{fin.}\,5\,\omega}{1 - x \cos.\,5\,\omega}:$$

eft vero $\text{fin.}\,\omega = \frac{1}{2}$; $\text{cof.}\,\omega = \frac{\sqrt{3}}{2}$; $\text{fin.}\,3\,\omega = 1$; $\text{cof.}\,3\,\omega = 0$; $\text{fin.}\,5\,\omega = \frac{1}{2}$; $\text{cof.}\,5\,\omega = -\frac{\sqrt{3}}{2}$, ergo

$$y = \tfrac{1}{3} \text{Arc. tang.} \frac{x}{2 - x\sqrt{3}} + \tfrac{2}{3} \text{Arc. tang.}\,x + \tfrac{1}{3} \text{Arc. tang.} \frac{x}{2 + x\sqrt{3}}:$$

feu

$$y = \tfrac{1}{3} \text{Arc. tang.} \frac{x}{1 - xx} + \tfrac{2}{3} \text{Arc. tang.}\,x = \tfrac{1}{3} \text{Arc. tang.} \frac{3x(1 - xx)}{3 - 4xx + x^4}$$

Corollarium 1.

137. Sit $z = \int \frac{xx\,\partial x}{1 + x^6} = \frac{1}{3} x^3 - \frac{1}{9} x^9 + \frac{1}{15} x^{15} - \frac{1}{21} x^{21} + \text{etc.}$ at facto $x^3 = u$, eft

$$z = \tfrac{1}{3} \int \frac{\partial u}{1 + uu} = \tfrac{1}{3} \text{ Arc. tang.}\,u = \tfrac{1}{3} \text{ Arc. tang.}\,x^3.$$

Hinc feries huiusmodi mixta formatur:

$$x+\tfrac{n}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7-\tfrac{n}{9}x^9-\tfrac{1}{11}x^{11}+\tfrac{1}{13}x^{13}+\tfrac{n}{15}x^{15}+\tfrac{1}{17}x^{17}-\text{etc.}$$

cuius fumma eft $=\tfrac{1}{3}$ Arc.tang. $\frac{3x(1-xx)}{1-4xx+x^4}+\tfrac{n}{3}$ Arc. tang. x^3.

Corollarium 2.

138. Si hic capiatur $n=-1$, binos angulos in vnum colligendo, fit

$$\tfrac{1}{3}\text{ Arc. tang. }\frac{3x(1-xx)}{1-4xx+x^4}-\tfrac{1}{3}\text{ Arc. tang. }x^3$$
$$=\tfrac{1}{3}\text{ Arc. tang. }\frac{3x-4x^3+4x^5-x^7}{1-4xx+4x^4-3x^6}:$$

quae fractio per $1-xx+x^4$ diuidendo, reducitur ad $\frac{3x-x^3}{1-3xx}$, quae eft tang. tripli anguli x pro tangente habentis, ita vt fit $\tfrac{1}{3}$ Arc. tang. $\frac{3x-x^3}{1-3xx}=$ Arc. tang. x, quod idem feries inuenta manifefto indicat.

Exemplum 6.

139. *Hanc formulam* $\partial y=\frac{(x^{m-1}+x^{n-m-1})\partial x}{1+x^n}$, *per feriem integrare.*

Ob $\frac{1}{1+x^n}=1-x^n+x^{2n}-x^{3n}+x^{4n}-\text{etc.}$ habebitur

$$y=\frac{x^m}{m}+\frac{x^{n-m}}{n-m}-\frac{x^{n+m}}{n+m}-\frac{x^{2n-m}}{2n-m}+\frac{x^{2n+m}}{2n+m}+\frac{x^{3n-m}}{3n-m}-\text{etc.}$$

Haec ergo feries per §. 82. aggregatum aliquot arcuum circularium exprimit, quos ibi videre licet.

Corollarium.

140. Eodem modo propofita formula $\partial z=\frac{(x^{m-1}-x^{n-m-1})\partial x}{1-x^n}$,

ob

ob $\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \text{etc.}$ inuenitur:

$$z = \frac{x^m}{m} - \frac{x^{n-m}}{n-m} + \frac{x^{n+m}}{n+m} - \frac{x^{2n-m}}{2n-m} + \frac{x^{2n+m}}{2n+m} - \frac{x^{3n-m}}{3n-m} + \text{etc.}$$

cuius valor §. 84. est exhibitus.

Exemplum 7.

141. *Hanc formulam* $\partial y = \frac{(1+2x)\partial x}{1+x+xx}$, *per seriem integrare.*

Primo integrale est manifesto $y = l(1+x+xx)$; vt autem in seriem conuertatur, multiplicetur numerator et denominator per $1-x$, vt fiat $\partial y = \frac{(1+x-2xx)\partial x}{1-x^3}$. Cum nunc sit $\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + x^{12} + \text{etc.}$ erit integrando:

$$y = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{2x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} - \frac{2x^9}{9} + \text{etc.}$$

Corollarium 1.

142. Eodem modo inueniri potest

$$y = l(1 + x + xx + x^3)$$

per seriem. Cum enim fiat $y + l(1-x) = l(1-x^4)$, erit

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \text{etc.}$$
$$- x^4 \qquad\qquad\qquad - \frac{x^8}{2}$$

siue

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3x^8}{8} + \frac{x^9}{9} + \text{etc.}$$

Corollarium 2.

143. At fractio $\frac{1+2x}{1+x+xx}$ per seriem recurrentem euoluta dat

 $1 + x$

$$1 + x - 2xx + x^3 + x^4 - 2x^5 + x^6 + x^7 - 2x^8 + \text{etc.}$$

vnde per integrationem eadem feries obtinetur, quae ante.

Exemplum 8.

144. *Hanc formulam* $\partial y = \frac{\partial x}{1 - 2x\,\text{cof.}\,\zeta + xx}$ *per feriem integrare.*

Per §. 64. vbi $A = 1$, $B = 0$, $a = 1$, et $b = 1$, eft huius formulae integrale $y = \frac{1}{\text{fin.}\,\zeta}$ Arc. tang. $\frac{x\,\text{fin.}\,\zeta}{1 - x\,\text{cof.}\,\zeta}$. At per feriem recurrentem reperimus

$$\frac{1}{1 - 2x\,\text{cof.}\,\zeta + xx} = 1 + 2x\,\text{cof.}\,\zeta + (4\,\text{cof.}\,\zeta^2 - 1)\,xx$$
$$+ (8\,\text{cof.}\,\zeta^3 - 4\,\text{cof.}\,\zeta)\,x^3 + (16\,\text{cof.}\,\zeta^4 - 12\,\text{cof.}\,\zeta^2 + 1)\,x^4$$
$$+ (32\,\text{cof.}\,\zeta^5 - 32\,\text{cof.}\,\zeta^3 + 6\,\text{cof.}\,\zeta)\,x^5 + \text{etc.}$$

qua ferie per ∂x multiplicata et integrata, obtinetur quaefitum. Poteftatibus autem ipfius cof. ζ, in cofinus angulorum multiplorum conuerfis reperitur:

$$y = x + \tfrac{1}{2}xx\,(2\,\text{cof.}\,\zeta) + \tfrac{1}{3}x^3\,(2\,\text{cof.}\,2\zeta + 1)$$
$$+ \tfrac{1}{4}x^4\,(2\,\text{cof.}\,3\zeta + 2\,\text{cof.}\,\zeta) + \tfrac{1}{5}x^5\,(2\,\text{cof.}\,4\zeta + 2\,\text{cof.}\,2\zeta + 1)$$
$$+ \tfrac{1}{6}x^6\,(2\,\text{cof.}\,5\zeta + 2\,\text{cof.}\,3\zeta + 2\,\text{cof.}\,\zeta)\ \text{etc.}$$

Corollarium 1.

145. Si ponatur $\partial z = \frac{(1 - x\,\text{cof.}\,\zeta)\,\partial x}{1 - 2x\,\text{cof.}\,\zeta + xx}$, erit per (§. 63.) $A = 1$, $B = -\text{cof.}\,\zeta$, $a = 1$ et $b = 1$, ideoque

$$z = -\text{cof.}\,\zeta\, l\sqrt{(1 - 2x\,\text{cof.}\,\zeta + xx)} + \text{fin.}\,\zeta\ \text{Arc. tang.}\ \frac{x\,\text{fin.}\,\zeta}{1 - x\,\text{cof.}\,\zeta}.$$

At per feriem

$$\text{ob}\ \frac{1 - x\,\text{cof.}\,\zeta}{1 - 2x\,\text{cof.}\,\zeta + xx} = 1 + x\,\text{cof.}\,\zeta + x^2\,\text{cof.}\,2\zeta$$
$$+ x^3\,\text{cof.}\,3\zeta + x^4\,\text{cof.}\,4\zeta + \text{etc.}\ \text{fit}$$
$$z = x + \tfrac{1}{2}xx\,\text{cof.}\,\zeta + \tfrac{1}{3}x^3\,\text{cof.}\,2\zeta + \tfrac{1}{4}x^4\,\text{cof.}\,3\zeta$$
$$+ \tfrac{1}{5}x^5\,\text{cof.}\,4\zeta + \text{etc.}$$

Corollarium 2.

146. At quia $\partial z = \frac{\partial x(-x\,cof.\,\zeta + cof.\,\zeta^2 + fin.\,\zeta^2)}{1 - 2x\,cof.\,\zeta + xx}$, erit

$$z = -cof.\zeta\, l\sqrt{(1-2x\,cof.\zeta + xx)} + fin.\zeta^2 \int \frac{\partial x}{1-2x\,cof.\zeta + xx}.$$

Hinc ergo pro $y = \int \frac{\partial x}{1-2x\,cof.\zeta+xx}$ alia reperitur feries infinita cum logarithmo connexa, fcilicet

$$y = \frac{cof.\zeta}{fin.\zeta^2}\, l\sqrt{(1-2x\,cof.\zeta + xx)}$$
$$+ \frac{1}{fin.\zeta^2}(x + \tfrac{1}{2}xx\,cof.\zeta + \tfrac{1}{3}x^3\,cof.2\zeta + \tfrac{1}{4}x^4\,cof.3\zeta + etc.)$$

Problema 12.

147. Formulam differentialem irrationalem $\partial y = x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}}$ per feriem infinitam integrare.

Solutio.

Sit $a^{\frac{\mu}{\nu}} = c$, erit $\partial y = c\,x^{m-1}\partial x(1+\frac{b}{a}x^n)^{\frac{\mu}{\nu}}$, vbi quidem affumimus c non effe quantitatem imaginariam. Cum igitur fit

$$(1+\tfrac{b}{a}x^n)^{\frac{\mu}{\nu}} = 1 + \frac{\mu b}{1\nu.a}x^n + \frac{\mu(\mu-\nu)bb}{1\nu.2\nu.aa}x^{2n} + \frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu.2\nu.3\nu.a^3}x^{3n} + etc.$$

erit integrando:

$$y = c\left(\frac{x^m}{m} + \frac{\mu b}{\nu.a}\cdot\frac{x^{m+n}}{m+n} + \frac{\mu(\mu-\nu)bb}{1\nu.2\nu.aa}\cdot\frac{x^{m+2n}}{m+2n}\right.$$
$$\left. + \frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu.2\nu.3\nu.a^3}\cdot\frac{x^{m+3n}}{m+3n} + etc.\right),$$

quae feries in infinitum excurrit, nifi $\frac{\mu}{\nu}$ fit numerus integer pofitiuus.

Sin autem casu, quo ν numerus par, a fuerit quantitas negatiua, expressio nostra ita est repraesentanda,

$$\partial y = x^{m-1}\partial x\,(bx^n - a)^{\frac{\mu}{\nu}} = b^{\frac{\mu}{\nu}}\, x^{m+\frac{\mu n}{\nu}-1}\partial x\,(1-\tfrac{a}{b}x^{-n})^{\frac{\mu}{\nu}}.$$

Cum igitur sit

$$(1-\tfrac{a}{b}x^{-n})^{\frac{\mu}{\nu}} = 1 - \frac{\mu a}{1\nu . b}x^{-n} + \frac{\mu(\mu-\nu)a^2}{1\nu . 2\nu . b^2}x^{-2n} - \frac{\mu(\mu-\nu)(\mu-2\nu)a^3}{1\nu . 2\nu . 3\nu . b^3}x^{-3n} + \text{etc.}$$

erit integrando

$$y = b^{\frac{\mu}{\nu}}\left(\frac{\nu x^{m+\frac{\mu n}{\nu}}}{m\nu+\mu n} - \frac{\mu a}{1\nu . b}\cdot\frac{\nu x^{m+\frac{(\mu-\nu)n}{\nu}}}{m\nu+(\mu-\nu)n} + \frac{\mu(\mu-\nu)a^2}{1\nu . 2\nu . b^2}\cdot\frac{\nu x^{m+\frac{(\mu-2\nu)n}{\nu}}}{m\nu+(\mu-2\nu)n} - \text{etc.}\right)$$

Si a et b sint numeri positiui, vtraque euolutione vti licet.

Exemplum I.

148. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(1-xx)}}$, *per seriem integrare.*

Primo ex superioribus patet esse $y = \text{Arc. sin. } x$ qui ergo angulus etiam per seriem infinitam exprimetur. Cum enim sit

$$\frac{1}{\sqrt{(1-xx)}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \frac{1.3.5.7}{2.4.6.8}x^8 + \text{etc.}$$

erit

$$y = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \frac{1.3.5.7}{2.4.6.8}\cdot\frac{x^9}{9} + \text{etc.}$$

vtroque valore ita definito, vt euanescat posito $x = 0$.

Corollarium I.

149. Si ergo sit $x = 1$, ob Arc. sin. $1 = \frac{\pi}{2}$, erit

$$\frac{\pi}{2} = 1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \frac{1.3.5}{2.4.6.7} + \frac{1.3.5.7}{2.4.6.8.9} + \text{etc.}$$

At

At si ponatur $x = \frac{1}{2}$, ob Arc. sin. $\frac{1}{2} = 30^\circ = \frac{\pi}{6}$, erit

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2.2^3.3} + \frac{1.3}{2.4.2^5.5} + \frac{1.3.5}{2.4.6.2^7.7} + \frac{1.3.5.7}{2.4.6.8.2^9.9} + \text{etc.}$$

cuius seriei decem termini additi dant 0,52359877, cuius sextuplum 3,14159262 tantum in octaua figura a veritate discrepat.

Corollarium 2.

150. Proposita hac formula $\partial y = \frac{\partial x}{\sqrt{(x - xx)}}$, posito $x = uu$, fit

$$\partial y = \frac{2u\partial u}{\sqrt{(uu - u^4)}} = \frac{2\partial u}{\sqrt{(1 - uu)}}:$$

ergo $y = 2$ Arc. sin. $u = 2$ Arc. sin. $\sqrt{x}$. Tum vero per seriem erit:

$$y = 2\left(u + \frac{1}{2}.\frac{u^3}{3} + \frac{1.3}{2.4}.\frac{u^5}{5} + \frac{1.3.5}{2.4.6}.\frac{u^7}{7} + \text{etc.}\right) \text{ seu}$$

$$y = 2\left(1 + \frac{1}{2}.\frac{x}{3} + \frac{1.3}{2.4}.\frac{xx}{5} + \frac{1.3.5}{2.4.6}.\frac{x^3}{7} + \text{etc.}\right)\sqrt{x}.$$

Exemplum 2.

151. *Formulam $\partial y = \partial x\sqrt{(2ax - xx)}$ per seriem integrare.*

Posito $x = uu$, fit $\partial y = 2uu\partial u\sqrt{(2a - uu)}$: at per reductionem I. (§. 118.) est $n = 2$; $m = 1$; $a = 2a$; $b = -1$; $\mu = 1$; $\nu = 2$; vnde

$$\int uu\partial u\sqrt{(2a-uu)} = -\frac{1}{4}u(2a-uu)^{\frac{3}{2}} + \frac{1}{2}a\int\partial u\sqrt{(2a-uu)}:$$

et per tertiam, sumendo $m = 1$, $a = 2a$, $b = -1$, $n = 2$, $\mu = -1$, $\nu = 2$, fit

$$\int\partial u\sqrt{(2a-uu)} = \frac{1}{2}u\sqrt{(2a-uu)} + a\int\frac{\partial u}{\sqrt{(2a-uu)}}:$$

at est

$$\int\frac{\partial u}{\sqrt{(2a-uu)}} = \text{Arc. sin. }\frac{u}{\sqrt{2a}} = \text{Arc. sin. }\frac{\sqrt{x}}{\sqrt{2a}}, \text{ ideoque}$$

$$\int uu\partial u\sqrt{(2a-uu)} = -\frac{1}{4}u(2a-uu)^{\frac{3}{2}} + \frac{1}{4}au\sqrt{(2a-uu)} + \frac{1}{2}aa\,\text{Arc.sin.}\frac{\sqrt{x}}{\sqrt{2a}}$$

$$= \frac{1}{4}u(uu-a)\sqrt{(2a-uu)} + \frac{1}{2}aa\,\text{Arc. sin. }\frac{\sqrt{x}}{\sqrt{2a}}.$$

Ergo

Ergo $$y = \tfrac{1}{2}(x-a)\sqrt{(2ax-xx)} + aa\,\text{Arc.fin.}\,\tfrac{\sqrt{x}}{\sqrt{2a}}.$$

Pro ferie autem inuenienda eft $\partial y = \partial x \sqrt{2ax}\,(1-\tfrac{x}{2a})^{\frac{1}{2}}$

$$= x^{\frac{1}{2}}\partial x\,(1-\tfrac{1}{2}\cdot\tfrac{x}{2a}-\tfrac{1.1}{2.4}\cdot\tfrac{xx}{4aa}-\tfrac{1.1.3}{2.4.6}\cdot\tfrac{x^3}{8a^3}-\text{etc.})\sqrt{2a}:$$

hincque integrando:

$$y = (\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{1}{2}\cdot\frac{2x^{\frac{5}{2}}}{5.2a}-\frac{1.1}{2.4}\cdot\frac{2x^{\frac{7}{2}}}{7.4aa}-\frac{1.1.3}{2.4.6}\cdot\frac{2x^{\frac{9}{2}}}{9.8a^3}-\text{etc.})\sqrt{2a},$$

feu

$$y = (\tfrac{x}{3}-\tfrac{1}{2}\cdot\tfrac{xx}{5.2a}-\tfrac{1.1}{2.4}\cdot\tfrac{x^3}{7.4aa}-\tfrac{1.1.3}{2.4.6}\cdot\tfrac{x^4}{9.8a^3}-\text{etc.})\,2\sqrt{2ax}.$$

Corollarium 1.

152. Integrale facilius inueniri poteft, ponendo $x = a-v$, vnde fit $\partial y = -\partial v\sqrt{(aa-vv)}$, et per reductionem tertiam

$$\int\partial v\sqrt{(aa-vv)} = \tfrac{1}{2}v\sqrt{(aa-vv)}+\tfrac{1}{2}aa\int\tfrac{\partial v}{\sqrt{(aa-vv)}},\ \text{hinc}$$

$$y = C-\tfrac{1}{2}v\sqrt{(aa-vv)}-\tfrac{1}{2}aa\,\text{Arc.fin.}\,\tfrac{v}{a},\ \text{feu}$$

$$y = C-\tfrac{1}{2}(a-x)\sqrt{(2ax-xx)}-\tfrac{1}{2}aa\,\text{Arc.fin.}\,\tfrac{a-x}{a}.$$

Vt igitur fiat $y=0$, pofito $x=0$, capi debet $C=\tfrac{1}{2}aa\,\text{Arc.fin.}\,1$, ita vt fit

$$y = -\tfrac{1}{2}(a-x)\sqrt{(2ax-xx)}+\tfrac{1}{2}aa\,\text{Arc.cof.}\,\tfrac{a-x}{a}.$$

Eft vero

$$\text{Arc.fin.}\,\tfrac{\sqrt{x}}{\sqrt{2a}} = \tfrac{1}{2}\,\text{Arc.cof.}\,\tfrac{a-x}{a}.$$

Corollarium 2.

153. Si ponamus $x=\tfrac{a}{2}$, fit $y = \tfrac{-aa\sqrt{3}}{8}+\tfrac{\pi aa}{6}$, feries autem dat

$$y = 2aa\,(\tfrac{1}{2.3}-\tfrac{1}{2.5.2^3}-\tfrac{1.1}{2.4.7.2^5}-\tfrac{1.1.3}{2.4.6.9.2^7}-\text{etc.})$$

vnde

vnde colligitur

$$\pi = \frac{3\sqrt{3}}{4} + 6\left(\frac{1}{3} - \frac{1}{2.5.2^2} - \frac{1.1}{2.4.7.2^4} - \frac{1.1.3}{2.4.6.9.2^6} - \text{etc.}\right)$$

at per ſuperiorem eſt

$$\pi = 3\left(1 + \frac{1}{2.3.2^2} + \frac{1.3}{2.4.5.2^4} + \frac{1.3.5}{2.4.6.7.2^6}\right)$$

ex quarum combinatione plures aliae formari poſſunt.

Exemplum 3.

154. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(1+xx)}}$, *per ſeriem integrare.*

Integrale eſt $y = l[x + \sqrt{(1 + xx)}]$, ita ſumtum vt euaneſcat poſito $x = 0$. At ob

$$\frac{1}{\sqrt{(1+xx)}} = 1 - \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 - \frac{1.3.5}{2.4.6}x^6 + \text{etc.}$$

erit idem integrale per ſeriem expreſſum:

$$y = x - \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} - \frac{1.3.5}{2.4.6}.\frac{x^7}{7} + \text{etc.}$$

Exemplum 4.

155. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(xx-1)}}$ *per ſeriem integrare.*

Integratio dat $y = l[x + \sqrt{(xx - 1)}]$ quod euaneſcit poſito $x = 1$. Iam ob

$$\frac{1}{\sqrt{(xx-1)}} = \frac{1}{x} + \frac{1}{2x^3} + \frac{1.3}{2.4.x^5} + \frac{1.3.5}{2.4.6x^7} \text{ etc.}$$

erit idem integrale:

$$y = C + lx - \frac{1}{2.2x^2} - \frac{1.3}{2.4.4x^4} - \frac{1.3.5}{2.4.6.6x^6} - \text{etc.}$$

quod vt euaneſcat poſito $x = 1$, conſtans ita definitur, vt fiat:

$$y = lx + \frac{1}{2.2}\left(1 - \frac{1}{xx}\right) + \frac{1.3}{2.4.4}\left(1 - \frac{1}{x^4}\right) + \frac{1.3.5}{2.4.6.6}\left(1 - \frac{1}{x^6}\right) + \text{etc.}$$

Corollarium.

156. Poſito $x = 1 + u$ fit

$$\partial y = \frac{\partial u}{\sqrt{(2u+uu)}} = \frac{\partial u}{\sqrt{2u}}\left(1 + \frac{u}{2}\right)^{-\frac{1}{2}} = \frac{\partial u}{\sqrt{2u}}\left(1 - \frac{1}{2}.\frac{u}{2} + \frac{1.3}{2.4}.\frac{uu}{4} - \frac{1.3.5}{2.4.6}.\frac{u^3}{8} + \text{etc}\right)$$

 vnde

vnde integrando habebitur:

$$y=\frac{1}{\sqrt{2}}\left(2\sqrt{u}-\frac{1}{2}\cdot\frac{2u^{\frac{3}{2}}}{2\cdot 3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{2u^{\frac{5}{2}}}{5\cdot 4}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{2u^{\frac{7}{2}}}{7\cdot 8}+\text{etc.}\right) \text{ seu}$$

$$y=\left(1-\frac{1u}{2\cdot 3\cdot 2}+\frac{1\cdot 3\cdot uu}{2\cdot 4\cdot 5\cdot 4}-\frac{1\cdot 3\cdot 5\cdot u^3}{2\cdot 4\cdot 6\cdot 7\cdot 8}+\text{etc.}\right)\sqrt{2u}.$$

Exemplum 5.

157. *Formulam* $\partial y=\frac{\partial x}{(1-x)^n}$ *per seriem integrare.*

Per integrationem fit

$$y=\frac{1}{(n-1)(1-x)^{n-1}}-\frac{1}{n-1},$$

facto $y=0$ si $x=0$, seu

$$y=\frac{(1-x)^{-n+1}-1}{n-1}.$$

Iam vero per seriem est

$$\partial y=\partial x\left(1+nx+\frac{n(n+1)}{1\cdot 2}x^2+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3+\text{etc.}\right)$$

vnde idem integrale ita exprimetur:

$$y=x+\frac{nx^2}{2}+\frac{n(n+1)x^3}{1\cdot 2\cdot 3}+\frac{n(n+1)(n+2)x^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}$$

Hinc autem quoque manifesto fit

$$(n-1)y+1=\frac{1}{(1-x)^{n-1}}.$$

Scholion.

158. Haec autem cum sint nimis obuia, quam vt iis fusius inhaerere sit opus, aliam methodum series eliciendi exponam magis absconditam, quae saepe in Analysi eximium vsum afferre potest.

Pro-

Problema 13.

159. Propoſita formula differentiali

$$\partial y = x^{m-1} \partial x (a + b x^n)^{\frac{\mu}{\nu} - 1},$$

eius integrale altera methodo in ſeriem conuertere.

Solutio.

Ponatur $y = (a + b x^n)^{\frac{\mu}{\nu}} z$, erit

$$\partial y = (a + b x^n)^{\frac{\mu}{\nu} - 1} [\partial z (a + b x^n) + \tfrac{n\mu}{\nu} b x^{n-1} z \partial x]:$$

vnde fit

$$x^{m-1} \partial x = \partial z (a + b x^n) + \tfrac{n\mu}{\nu} b x^{n-1} z \partial x, \text{ ſeu}$$

$$\nu x^{m-1} \partial x = \nu \partial z (a + b x^n) + n \mu b x^{n-1} z \partial x.$$

Iam antequam ſeriem, qua valor ipſius z definiatur, inueſtigemus, notandum eſt caſu, quo x euaneſcit, fieri

$$\partial y = a^{\frac{\mu}{\nu} - 1} x^{m-1} \partial x = a^{\frac{\mu}{\nu}} \partial z,$$

vt ſit $\partial z = \frac{1}{a} x^{m-1} \partial x$. Statuamus ergo:

$$z = \mathrm{A} x^m + \mathrm{B} x^{m+n} + \mathrm{C} x^{m+2n} + \mathrm{D} x^{m+3n} + \text{etc.}$$

eritque

$$\tfrac{\partial z}{\partial x} = m \mathrm{A} x^{m-1} + (m+n) \mathrm{B} x^{m+n-1} + (m+2n) \mathrm{C} x^{m+2n-1} + \text{etc.}$$

Subſtituantur hae ſeries loco z et $\frac{\partial z}{\partial x}$ in aequatione

$$\tfrac{\nu \partial z}{\partial x} (a + b x^n) + n \mu b x^{n-1} z - \nu x^{m-1} = 0,$$

ſingulisque terminis ſecundum poteſtates ipſius x diſpoſitis, orietur iſta aequatio:

$$\left.\begin{array}{llll} m\nu a \mathrm{A} x^{m-1} & + (m+n)\nu a \mathrm{B} x^{m+n-1} & + (m+2n)\nu a \mathrm{C} x^{m+2n-1} & + \text{etc.} \\ -\nu & + m\nu b \mathrm{A} & + (m+n)\nu b \mathrm{B} & \\ & + n\mu b \mathrm{A} & + n\mu b \mathrm{B} & \end{array}\right\} = 0:$$

vnde ſingulis terminis nihilo aequalibus poſitis, coefficientes

 ficti

ficti per sequentes formulas definientur:

$$m\nu a\mathrm{A}-\nu=0;\qquad \text{hinc}\quad \mathrm{A}=\frac{1}{ma};$$

$$(m+n)\nu a\mathrm{B}+(m\nu+n\mu)b\mathrm{A}=0;\quad \mathrm{B}=-\frac{(m\nu+n\mu)b}{(m+n)\nu a}\mathrm{A};$$

$$(m+2n)\nu a\mathrm{C}+[(m+n)\nu+n\mu]b\mathrm{B}=0;\quad \mathrm{C}=-\frac{[(m+n)\nu+n\mu]b}{(m+2n)\nu a}\mathrm{B};$$

$$(m+3n)\nu a\mathrm{D}+[(m+2n)\nu+n\mu]b\mathrm{C}=0;\quad \mathrm{D}=-\frac{[(m+2n)\nu+n\mu]b}{(m+3n)\nu a}\mathrm{C};$$

sicque quilibet coefficiens facile ex praecedente reperitur. Tum vero erit:

$$y=(a+bx^n)^{\frac{\mu}{\nu}}(\mathrm{A}x^m+\mathrm{B}x^{m+n}+\mathrm{C}x^{m+2n}+\mathrm{D}x^{m+3n}+\text{etc.})$$

Solutio 2.

Quemadmodum hic seriem secundum potestates ipsius x ascendentem assumsimus, ita etiam descendentem constituere licet:

$$z=\mathrm{A}x^{m-n}+\mathrm{B}x^{m-2n}+\mathrm{C}x^{m-3n}+\mathrm{D}x^{m-4n}+\text{etc.}$$

vt sit

$$\frac{\partial z}{\partial x}=(m-n)\mathrm{A}x^{m-n-1}+(m-2n)\mathrm{B}x^{m-2n-1}+(m-3n)\mathrm{C}x^{m-3n-1}\ \text{etc.}$$

quibus seriebus substitutis prodit:

$$\left.\begin{array}{llll}
+(m-n)\nu b\mathrm{A}x^{m-1} & +(m-n)\nu a\mathrm{A}x^{m-n-1} & +(m-2n)\nu a\mathrm{B}x^{m-2n-1} & +(m-3n)\nu a\mathrm{C}x^{m-3n-1} \\
+n\mu b\mathrm{A} & +(m-2n)\nu b\mathrm{B} & +(m-3n)\nu b\mathrm{C} & +(m-4n)\nu b\mathrm{D} \\
-\nu & +n\mu b\mathrm{B} & +n\mu b\mathrm{C} & +n\mu b\mathrm{D}
\end{array}\right\}=0.$$

Hinc ergo sequenti modo litterae A, B, C, etc. determinantur:

$$(m-n)\nu b\mathrm{A}+n\mu b\mathrm{A}-\nu=0\quad \text{ergo}\quad \mathrm{A}=\frac{\nu}{(m-n)\nu+n\mu}\cdot\frac{1}{b};$$

$$(m-n)\nu a\mathrm{A}+(m-2n)\nu b\mathrm{B}+n\mu b\mathrm{B}=0,\quad \mathrm{B}=\frac{-(m-n)\nu}{(m-2n)\nu+n\mu}\cdot\frac{a}{b}\mathrm{A};$$

$$(m-2n)\nu a\mathrm{B}+(m-3n)\nu b\mathrm{C}+n\mu b\mathrm{C}=0,\quad \mathrm{C}=\frac{-(m-2n)\nu}{(m-3n)\nu+n\mu}\cdot\frac{a}{b}\mathrm{B};$$

$$(m-3n)\nu a\mathrm{C}+(m-4n)\nu b\mathrm{D}+n\mu b\mathrm{D}=0,\quad \mathrm{D}=\frac{-(m-3n)\nu}{(m-4n)\nu+n\mu}\cdot\frac{a}{b}\mathrm{C};$$

vbi iterum lex progressionis harum litterarum est manifesta.

Co-

Corollarium 1.

160. Prior feries ideo eft memorabilis, quod cafibus, quibus $(m+in)\nu+n\mu=0$, feu $-\frac{m}{n}-\frac{\mu}{\nu}=i$, abrumpitur, atque ipfum integrale algebraicum exhibet. Pofterior vero abrumpitur, quoties $m-in=0$ feu $\frac{m}{n}=i$, denotante i numerum integrum pofitiuum.

Corollarium 2.

161. Vtraque vero feries etiam incommodo quodam laborat, quod non femper in vfum vocari poteft. Quando enim vel $m=0$, vel $m+in=0$, priori vti non licet: quando vero $(m-in)\nu+n\mu=0$, feu $\frac{m}{n}+\frac{\mu}{\nu}=i$, vfus pofterioris tollitur, quia termini fierent infiniti.

Corollarium 3.

162. Hoc vero commode vfu venit, vt quoties altera applicari nequit, altera certo in vfum vocari poffit, iis tantum cafibus exceptis, quibus et $-\frac{m}{n}$ et $\frac{\mu}{\nu}+\frac{m}{n}$ funt numeri integri pofitiui. Quia autem tum eft $\nu=1$, hi cafus funt rationales integri, nihilque difficultatis habent.

Corollarium 4.

163. Poffunt etiam ambae feries fimul pro z coniungi hoc modo: Sit prior feries $=P$, pofterior vero $=Q$, vt capi poffit tam $z=P$, quam $z=Q$. Binis autem coniungendis, erit $z=\alpha P+\beta Q$, dummodo fit $\alpha+\beta=1$.

Scholion.

164. Inde autem, quod duas feries pro z exhibemus, minime fequitur, has duas feries inter fe effe aequales, neque enim neceffe eft, vt valores ipfius y inde orti fiant aequales, dummodo quantitate conftante a fe inuicem differant. Ita fi prior feries inuenta per P, pofterior per Q indicetur, quia ex

illa fit $y = (a + b x^n)^{\frac{\mu}{\nu}} P$, ex hac vero $y = (a + b x^n)^{\frac{\mu}{\nu}} Q$, certo erit $(a + b x^n)^{\frac{\mu}{\nu}} (P - Q)$ quantitas conſtans, ideoque $P - Q = C (a + b x^n)^{-\frac{\mu}{\nu}}$. Vtraque ſcilicet ſeries tantum integrale particulare praebet, quoniam nullam conſtantem inuoluit, quae non iam in formula differentiali contineatur. Interim tamen eadem methodo etiam valor completus pro z erui poteſt: praeter ſeriem enim aſſumtam P vel Q ſtatui poteſt

$$z = P + \alpha + \beta x^n + \gamma x^{2n} + \delta x^{3n} + \varepsilon x^{4n} + \text{etc.}$$

ac ſubſtitutione facta, ſeries P vt ante definitur, pro altera vero noua ſerie efficiendum eſt, vt ſit

$$\left.\begin{array}{llll} n\nu a\beta x^{n-1} & + 2n\nu a\gamma x^{2n-1} & + 3n\nu a\delta x^{3n-1} & + 4n\nu a\varepsilon x^{4n-1} \\ + n\mu b\alpha & + n\nu b\beta & + 2n\nu b\gamma & + 3n\nu b\delta \\ & + n\mu b\beta & + n\mu b\gamma & + n\mu b\delta \end{array}\right\} = 0,$$

vnde ducuntur hae determinationes:

$$\beta = \frac{-\mu}{\nu} \cdot \frac{b}{a} \alpha; \quad \gamma = \frac{-(\mu+\nu) b}{2\nu a} \cdot \beta; \quad \delta = \frac{-(\mu+2\nu) b}{3\nu a} \cdot \gamma;$$

$$\varepsilon = \frac{-(\mu+3\nu) b}{4\nu a} \cdot \delta \text{ etc.}$$

ita vt prodeat

$$z = P + \alpha \left(1 - \frac{\mu}{\nu} \cdot \frac{b}{a} x^n + \frac{\mu(\mu+\nu)}{\nu \cdot 2\nu} \cdot \frac{b^2}{a^2} x^{2n} - \frac{\mu(\mu+\nu)(\mu+2\nu)}{\nu \cdot 2\nu \cdot 3\nu} \cdot \frac{b^3}{a^3} x^{3n} + \text{etc.}\right)$$

ſeu $z = P + \alpha \left(1 + \frac{b}{a} x^n\right)^{-\frac{\mu}{\nu}}$, hincque

$$y = P (a + b x^n)^{\frac{\mu}{\nu}} + \alpha a^{\frac{\mu}{\nu}};$$

quod eſt integrale completum quia conſtans α manſit arbitraria.

Exemplum 1.

165. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(1 - xx)}}$ *hoc modo per ſeriem integrare.*

Comparatione cum forma generali inſtituta, fit $a = 1$, $b = -1$, $m = 1$, $n = 2$, $\mu = 1$, $\nu = 2$: vnde poſito $y = z \surd$

$z\sqrt{(1-xx)}$ prima folutio

$$z = Ax + Bx^3 + Cx^5 + Dx^7 + \text{etc. praebet}$$

$$A = 1;\ B = \tfrac{2}{3}A;\ C = \tfrac{4}{5}B;\ D = \tfrac{6}{7}C;\ E = \tfrac{8}{9}D;\ \text{etc.}$$

vnde colligimus:

$$y = (x + \tfrac{2}{3}x^3 + \tfrac{2.4}{3.5}x^5 + \tfrac{2.4.6}{3.5.7}x^7 + \text{etc.})\sqrt{(1-xx)},$$

quod integrale euanefcit pofito $x = 0$, eft ergo $y = \text{Arc. fin.} x$. Altera methodus hic fruftra tentatur, ob $\frac{m}{n} + \frac{\mu}{\nu} = 1$.

Corollarium 1.

166. Pofito $x = 1$, videtur hinc fieri $y = 0$, ob $\sqrt{(1-xx)} = 0$: at perpendendum eft, fieri hoc cafu feriei infinitae fummam infinitam, ita vt nihil obftet, quo minus fit $y = \frac{\pi}{2}$. Si ponamus $x = \frac{1}{2}$, fit $y = 30° = \frac{\pi}{6}$, ideoque

$$\tfrac{\pi}{6} = (1 + \tfrac{2}{3.4} + \tfrac{2.4}{3.5.4^2} + \tfrac{2.4.6}{3.5.7.4^3} + \text{etc.})\tfrac{\sqrt{3}}{4}.$$

Corollarium 2.

167. Simili modo propofita formula $\partial y = \frac{\partial x}{\sqrt{(1+xx)}}$ reperitur:

$$y = (x - \tfrac{2}{3}x^3 + \tfrac{2.4}{3.5}x^5 - \tfrac{2.4.6}{3.5.7}x^7 + \text{etc.})\sqrt{(1+xx)}$$

eftque $y = l\,[x + \sqrt{(1+xx)}]$.

Exemplum 2.

168. *Formulam* $\partial y = \frac{\partial x}{x\sqrt{(1-xx)}}$ *hoc modo per feriem integrare.*

Eft ergo $m = 0$, $n = 2$, $\mu = 1$, $\nu = 2$, $a = 1$, et $b = -1$, vtendum igitur eft altera ferie fumendo

$$z = \frac{y}{\sqrt{(1-xx)}} = Ax^{-2} + Bx^{-4} + Cx^{-6} + Dx^{-8} + \text{etc.}$$

fitque

$$A = 1;\ B = \tfrac{2}{3}A;\ C = \tfrac{4}{5}B;\ D = \tfrac{6}{7}C;\ \text{etc.}$$

Hinc

Hinc ergo colligimus:

$$y = \left(\frac{1}{xx} + \frac{2}{3x^4} + \frac{2.4}{3.5x^6} + \frac{2.4.6}{3.5.7x^8} + \text{etc.}\right)\sqrt{(1-xx)}.$$

At integratio praebet $y = l\frac{1-\sqrt{(1-xx)}}{x}$, qui valores conueniunt, quia vterque euanescit posito $x = 1$.

Corollarium 1.

169. Cum autem haec series non conuergat nisi capiatur $x > 1$, hoc autem casu formula $\sqrt{(1-xx)}$ fiat imaginaria, haec series nullius est vsus.

Corollarium 2.

170. Si proponatur $\partial y = \frac{\partial x}{x\sqrt{(xx-1)}}$, eadem pro y series emergit per $\sqrt{-1}$ multiplicata, eritque

$$y = -\left(\frac{1}{xx} + \frac{2}{3x^3} + \frac{2.4}{3.5x^5} + \frac{2.4.6}{3.5.7x^7} + \text{etc.}\right)\sqrt{(xx-1)}.$$

Posito autem $x = \frac{1}{u}$, erit $\partial y = \frac{-\partial u}{\sqrt{(1-uu)}}$, et $y = C - \text{Arc. sin.}\, u$, seu $y = C - \text{Arc. sin.}\, \frac{1}{x}$: vbi sumi oportet $C = 0$, quia series illa euanescit posito $x = \infty$: ita vt sit $y = -\text{Arc. sin.}\, \frac{1}{x}$, quae cum superiori conuenit statuendo $\frac{1}{x} = v$.

Exemplum 3.

171. *Formulam $\partial y = \frac{\partial x}{\sqrt{(a+bx^4)}}$ hoc modo per seriem integrare.*

Est hic $m = 1$, $n = 4$, $\mu = 1$, $\nu = 2$, ideoque posito $y = z\sqrt{(a+bx^4)}$, prior resolutio dat

$$z = Ax + Bx^5 + Cx^9 + Dx^{13} + \text{etc.}$$

existente

$$A = \frac{1}{a};\ B = \frac{-3b}{5a}A;\ C = \frac{-7b}{9a}B;\ D = \frac{-11b}{13b}C \text{ etc.}$$

ita vt sit

$$y = \left(\frac{x}{a} - \frac{3bx^5}{5aa} + \frac{3.7b^2x^9}{5.9a^3} - \frac{3.7.11b^3x^{13}}{5.9.13a^4} + \text{etc.}\right)\sqrt{(a+bx^4)}.$$

Hic

Hic autem quoque altera resolutio locum habet, ponendo

$$z = A x^{-3} + B x^{-7} + C x^{-11} + D x^{-15} + \text{etc.}$$

existente

$$A = \tfrac{-1}{b};\ B = \tfrac{-3a}{5b} A;\ C = \tfrac{-7a}{9b} B;\ D = \tfrac{-11a}{13b} C;\ \text{etc.}$$

vnde colligitur:

$$y = -\left(\tfrac{1}{bx^3} - \tfrac{3a}{5b^2x^7} + \tfrac{3.7aa}{5.9b^3x^{11}} - \tfrac{3.7.11a^3}{5.9.13b^4x^{15}} + \text{etc.}\right)\sqrt{(a+bx^4)}$$

quarum serierum illa euanescit posito $x = 0$, haec vero posito $x = \infty$.

Corollarium 1.

172. Differentia ergo harum duarum serierum est constans, scilicet:

$$\left.\begin{matrix} +\tfrac{x}{a} - \tfrac{3bx^5}{5aa} + \tfrac{3.7b^2x^9}{5.9a^3} - \tfrac{3.7.11b^3x^{13}}{5.9.13a^4} + \text{etc.} \\ +\tfrac{1}{bx^3} - \tfrac{3a}{5bbx^7} + \tfrac{3.7a^2}{5.9b^3x^{11}} - \tfrac{3.7.11a^3}{5.9.13b^4x^{15}} + \text{etc.} \end{matrix}\right\}\sqrt{(a+bx^4)} = \text{Const.}$$

Corollarium 2.

173. Has ergo binas series colligendo habebimus

$$\tfrac{a+bx^4}{abx^3} - \tfrac{3}{5}\cdot\tfrac{a^3+b^3x^{12}}{a^2b^2x^7} + \tfrac{3.7}{5.9}\cdot\tfrac{a^5+b^5x^{20}}{a^3b^3x^{11}} - \text{etc.} = \tfrac{C}{\sqrt{(a+bx^4)}}$$

vbi quicunque valor ipsi x tribuatur, pro C semper eadem quantitas obtinetur.

Corollarium 3.

174. Ita si $a = 1$ et $b = 1$, erit haec series in $\sqrt{(1+x^4)}$ ducta semper constans, scilicet

$$\left(\tfrac{1+x^4}{x^3} - \tfrac{3}{5}\cdot\tfrac{1+x^{12}}{x^7} + \tfrac{3.7}{5.9}\cdot\tfrac{1+x^{20}}{x^{11}} - \text{etc.}\right)\sqrt{(1+x^4)} = C.$$

Cum igitur posito $x = 1$, fiat

$$C = \left(1 - \tfrac{3}{5} + \tfrac{3.7}{5.9} - \tfrac{3.7.11}{5.9.13} + \text{etc.}\right) 2\sqrt{2},$$

huicque valori etiam illa series, quicunque valor ipsi x tribuatur, est aequalis.

175. Haec poſtrema ſeries ſignis alternantibus procedens, per differentias facile in aliam iisdem ſignis praeditam transformatur, vnde eadem conſtaus concluditur

$$C = \left(1 + \frac{1}{3} + \frac{1.3}{5.9} + \frac{1.3.5}{5.9.13} + \frac{1.3.5.7}{5.9.13.17} + \text{etc.}\right)\sqrt{2},$$

quae ſeries ſatis cito conuergit, eritque proxime $C = \frac{13}{7}$.

Scholion.

176. Iſta methodus in hoc conſiſtit, vt ſeries quaedam indefinita fingatur, eiusque determinatio ex natura rei deriuetur. Eius vſus autem potiſſimum cernitur in aquationibus differentialibus reſoluendis; verum etiam in praeſenti inſtituto ſaepe vtiliter adhibetur. Eiusdem quoque methodi ope quantitates tranſcendentes reciprocae, veluti exponentiales et ſinus coſinusue angulorum, per ſeries exprimuntur, quae etſi iam aliunde ſint cognitae, tamen earum inueſtigationem per integrationem expoſuiſſe iuuabit, cum ſimili modo alia praeclara erui queant.

Problema 14.

177. Quantitatem exponentialem $y = a^x$ in ſeriem conuertere.

Solutio.

Sumtis logarithmis, habemus $l\,y = x\,l\,a$, et differentiando $\frac{\partial y}{y} = \partial x\,l\,a$, ſeu $\frac{\partial y}{\partial x} = y\,l\,a$: vnde valorem ipſius y per ſeriem quaeri oportet. Cum autem integrale completum latius pateat, notetur noſtro caſu poſito $x = 0$, fieri debere $y = 1$: quare fingatur haec pro y ſeries:

$$y = 1 + A\,x + B\,x^2 + C\,x^3 + D\,x^4 + \text{etc.}$$

vnde fit

$$\frac{\partial y}{\partial x} = A + 2\,B\,x + 3\,C\,x^2 + 4\,D\,x^3 + \text{etc.}$$

qui-

quibus ſubſtitutis in aequatione $\frac{\partial y}{\partial x} - y\,l\,a = 0$, erit

$$\left.\begin{array}{l} A + 2\,B\,x + 3\,C\,x^2 + 4\,D\,x^3 + 5\,E\,x^4 + \text{etc.} \\ -la - A\,l\,a - B\,l\,a \quad - C\,l\,a \quad - D\,l\,a \quad - \text{etc.}\end{array}\right\} = 0,$$

hincque coefficientes ita determinantur:

$$A = l\,a;\; B = \tfrac{1}{2}A\,l\,a;\; C = \tfrac{1}{3}B\,l\,a;\; D = \tfrac{1}{4}C\,l\,a \text{ etc.}$$

ſicque conſequimur:

$$y = a^x = 1 + \frac{x\,l\,a}{1} + \frac{x^2(l\,a)^2}{1.\,2} + \frac{x^3(l\,a)^3}{1.\,2.\,3} + \frac{x^4(l\,a)^4}{1.\,2.\,3.\,4} + \text{etc.}$$

quae eſt ipſa ſeries notiſſima in Introductione data.

Scholion.

178. Pro ſinibus et coſinibus angulorum ad differentialia ſecundi gradus eſt deſcendendum, ex quibus deinceps ſeries integrale referens elici debet. Cum autem gemina integratio duplicem determinationem requirat, ſeries ita eſt fingenda, vt duabus conditionibus ex natura rei petitis ſatisfaciat. Verum haec methodus etiam ad alias inueſtigationes extenditur, quae adeo in quantitatibus algebraicis verſantur, a cuiusmodi exemplo hic inchoemus.

Problema 15.

179. Hanc expreſſionem $y = [x + \sqrt{(1 + x\,x)}]^n$ in ſeriem ſecundum poteſtates ipſius x progredientem conuertere.

Solutio.

Quia eſt $l\,y = n\,l\,[x + \sqrt{(1 + x\,x)}]$ erit $\frac{\partial y}{y} = \frac{n\,\partial x}{\sqrt{(1 + x\,x)}}$; iam ad ſignum radicale tollendum ſumantur quadrata, erit $(1 + x\,x)\,\partial y^2 = n\,n\,y\,y\,\partial x^2$. Aequatio, ſumto ∂x conſtante, denuo differentietur, vt per $2\,\partial y$ diuiſo prodeat

$$\partial\partial y\,(1 + x\,x) + x\,\partial x\,\partial y - n\,n\,y\,\partial x^2 = 0:$$

vnde y per ſeriem elici debet. Primo autem patet, ſi ſit $x = 0$

 fore

fore $y = 1$, ac si x infinite paruum, $y = (1+x)^n = 1 + nx$. Fingatur ergo talis feries:

$$y = 1 + nx + Ax^2 + Bx^3 + Cx^4 + Dx^5 + Ex^6 + \text{etc.}$$

ex qua colligitur:

$$\frac{\partial y}{\partial x} = n + 2Ax + 3Bxx + 4Cx^3 + 5Dx^4 + 6Ex^5 + \text{etc. et}$$

$$\frac{\partial\partial y}{\partial x^2} = 2A + 6Bx + 12Cxx + 20Dx^3 + 30Ex^4 + \text{etc.}$$

Facta ergo fubftitutione adipifcimur:

$$\left.\begin{array}{ccccccc} 2A + 6Bx + & 12Cxx + & 20Dx^3 + & 30Ex^4 + & 42Fx^5 + & \text{etc.} \\ & + 2A & + 6B & + 12C & + 20D & + \text{etc.} \\ + nx + & 2A & + 3B & + 4C & + 5D & + \text{etc.} \\ - nn - n^3 - & An^2 & - Bn^2 & - Cn^2 & - Dn^2 & + \text{etc.} \end{array}\right\} = 0;$$

hincque deriuantur fequentes determinationes

$$A = \frac{nn}{2};\quad B = \frac{n(nn-1)}{2.3};\quad C = \frac{A(nn-4)}{3.4};\quad D = \frac{B(nn-9)}{4.5};\ \text{etc.}$$

ita vt fit

$$y = 1 + nx + \frac{nn}{1.2}x^2 + \frac{n(nn-1)}{1.2.3}x^3 + \frac{nn(nn-4)}{1.2.3.4}x^4 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5$$
$$+ \frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}x^6 + \frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7} + \text{etc.}$$

Corollarium 1.

180. Vti eft $y = [x + \sqrt{(1+xx)}]^n$, fi ftatuamus $z = [-x + \sqrt{(1+xx)}]^n$, pro z fimilis feries prodit, in qua x tantum negatiue capitur, hinc ergo concluditur:

$$\frac{y+z}{2} = 1 + \frac{nn}{1.2}x^2 + \frac{nn(nn-4)}{1.2.3.4}x^4 + \frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}x^6 + \text{etc. et}$$

$$\frac{y-z}{2} = nx + \frac{n(nn-1)}{1.2.3}x^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5$$
$$+ \frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7 + \text{etc.}$$

Corollarium 2.

181. Si ponatur $x = \sqrt{-1}.\text{fin.}\,\Phi$, erit $\sqrt{(1+xx)} = \text{cof.}\,\Phi$; hincque

$y =$

$$y = (\text{cof.}\,\Phi + \sqrt{-1}\,.\,\text{fin.}\,\Phi)^n = \text{cof.}\,n\,\Phi + \sqrt{-1}\,.\,\text{fin.}\,n\,\Phi, \text{ et}$$
$$z = (\text{cof.}\,\Phi - \sqrt{-1}\,.\,\text{fin.}\,\Phi)^n = \text{cof.}\,n\,\Phi - \sqrt{-1}\,.\,\text{fin.}\,n\,\Phi:$$

vnde deducimus:

$$\text{cof.}\,n\Phi = 1 - \frac{nn}{1.2}\,\text{fin.}\,\Phi^2 + \frac{nn(nn-4)}{1.2.3.4}\,\text{fin.}\,\Phi^4 - \frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}\,\text{fin.}\,\Phi^6 + \text{etc.}$$
$$\text{fin.}\,n\,\Phi = n\,\text{fin.}\,\Phi - \frac{n(nn-1)}{1.2.3}\,\text{fin.}\,\Phi^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}\,\text{fin.}\,\Phi^5$$
$$- \frac{n(nn-1)(nn-9)(nn-25)}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7}\,\text{fin.}\,\Phi^7 + \text{etc.}$$

Corollarium 3.

182. Hae feries ad multiplicationem angulorum pertinent, atque hoc habent fingulare, quod prior tantum cafibus, quibus n eft numerus par, pofterior vero, quibus eft numerus impar, abrumpatur.

Problema 16.

183. Propofito angulo Φ, tam eius finum quam cofinum per feriem infinitam exprimere.

Solutio.

Sit $y = \text{fin.}\,\Phi$ et $z = \text{cof.}\,\Phi$, erit $\partial y = \partial\Phi\sqrt{(1-yy)}$ et $\partial z = -\partial\Phi\sqrt{(1-zz)}$. Sumantur quadrata

$$\partial y^2 = \partial\Phi^2(1-yy) \text{ et } \partial z^2 = \partial\Phi^2(1-zz):$$

differentietur fumto $\partial\Phi$ conftante, fietque

$$\partial\partial y = -y\,\partial\Phi^2 \text{ et } \partial\partial z = -z\,\partial\Phi^2,$$

ficque y et z ex eadem aequatione difiniri oportet. Sed pro $y = \text{fin.}\,\Phi$ obferuandum eft, fi Φ euanefcat, fieri $y = \Phi$; pro $z = \text{cof.}\,\Phi$ verum, fi Φ euanefcat, fieri $z = 1 - \frac{1}{2}\Phi\Phi$, feu $z = 1 + 0\,\Phi$. Fingatur ergo

$$y = \Phi + A\,\Phi^3 + B\,\Phi^5 + C\,\Phi^7 + \text{etc.}$$
$$z = 1 + \alpha\,\Phi^2 + \beta\,\Phi^4 + \gamma\,\Phi^6 + \delta\,\Phi^8 + \text{etc.}$$

fietque fubftitutione facta:

$$\left.\begin{array}{l} 2\cdot 3\, A\Phi + 4\cdot 5\, B\Phi^3 + 6\cdot 7\, C\Phi^5 + \text{etc.} \\ +\quad 1 \quad +\quad A \quad +\quad B \end{array}\right\} = 0 \text{ et}$$

$$\left.\begin{array}{l} 1\cdot 2\, \alpha + 3\cdot 4\, \beta\Phi^2 + 5\cdot 6\, \gamma\Phi^4 + \text{etc.} \\ +\quad 1 \quad +\quad \alpha \quad +\quad \beta \end{array}\right\} = 0:$$

vnde colligimus:

$$A = \tfrac{-1}{2\cdot 3};\; B = \tfrac{-A}{4\cdot 5};\; C = \tfrac{-B}{6\cdot 7};\; D = \tfrac{-C}{8\cdot 9};\; \text{etc.}$$

$$\alpha = \tfrac{-1}{1\cdot 2};\; \beta = \tfrac{-\alpha}{3\cdot 4};\; \gamma = \tfrac{-\beta}{5\cdot 6};\; \delta = \tfrac{-\gamma}{7\cdot 8};\; \text{etc.}$$

vnde feries iam notiffimae obtinentur:

$$\text{fin.}\,\Phi = \tfrac{\Phi}{1} - \tfrac{\Phi^3}{1\cdot 2\cdot 3} + \tfrac{\Phi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \tfrac{\Phi^7}{1\cdot 2\cdot \ldots \cdot 7} + \text{etc.}$$

$$\text{cof.}\,\Phi = 1 - \tfrac{\Phi^2}{1\cdot 2} + \tfrac{\Phi^4}{1\cdot 2\cdot 3\cdot 4} - \tfrac{\Phi^6}{1\cdot 2\cdot \ldots \cdot 7} + \text{etc.}$$

Scholion.

184. Non opus erat ad differentialia fecundi gradus defcendere: fed ex formularum $y = \text{fin.}\,\Phi$ et $z = \text{cof.}\,\Phi$ differentialibus, quae funt $\partial y = z\,\partial\Phi$ et $\partial z = -y\,\partial\Phi$, eaedem feries facile reperiuntur. Fictis enim feriebus vt ante $y = \Phi + A\Phi^3 + B\Phi^5 + C\Phi^7$ etc. et $z = 1 + \alpha\Phi^2 + \beta\Phi^4 + \gamma\Phi^6$ + etc. fubftitutione facta, obtinebitur:

ex priore

$$\left.\begin{array}{l} 1 + 3\, A\Phi^2 + 5\, B\Phi^4 + 7\, C\Phi^6 + \text{etc.} \\ -1 - \quad \alpha \quad - \quad \beta \quad - \quad \gamma \end{array}\right\} = 0$$

ex pofteriore

$$\left.\begin{array}{l} 2\,\alpha\Phi + 4\,\beta\Phi^3 + 6\,\gamma\Phi^5 + \text{etc.} \\ +\quad 1 \quad +\quad A \quad +\quad B \end{array}\right\} = 0:$$

vnde colliguntur hae determinationes:

$$\alpha = -\tfrac{1}{2};\; A = \tfrac{\alpha}{3};\; \beta = \tfrac{-A}{4};\; B = \tfrac{\beta}{5};\; \gamma = \tfrac{-B}{6};\; C = \tfrac{\gamma}{7};\; \text{etc.}$$

ideoque

$$\alpha = -\tfrac{1}{2};\; \beta = +\tfrac{1}{2\cdot 3\cdot 4};\; \gamma = -\tfrac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6};\; \text{etc.}$$

A =

$$A = -\frac{1}{2.3}; \quad B = +\frac{1}{2.3.4.5}; \quad C = -\frac{1}{2.3.4.5.6.7}, \text{ etc.}$$

qui valores cum praecedentibus conueniunt. Hinc intelligitur, quomodo saepe duae aequationes simul facilius per series euoluuntur, quam si alteram seorsim tractare velimus.

Problema 17.

185. Per seriem exprimere valorem quantitatis y, qui satisfaciat huic aequationi $\frac{m\,\partial y}{\sqrt{(a+byy)}} = \frac{n\,\partial x}{\sqrt{(f+gxx)}}$.

Solutio.

Integratio huius aequationis suppeditat:

$$\frac{m}{\sqrt{b}}\, l\,[\sqrt{(a+byy)} + y\sqrt{b}] = \frac{n}{\sqrt{g}}\, l\,[\sqrt{(f+gxx)} + x\sqrt{g}] + C,$$

vnde deducimus:

$$y = \frac{1}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)} + x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}$$

$$-\frac{a}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)} - x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}};$$

constantes h et k ita capiendo, vt sit $hk = f$. Hinc discimus, si x sumatur euanescens, fore

$$y = \frac{1}{2\sqrt{b}}\left(\frac{\sqrt{f} + x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - \frac{a}{2\sqrt{b}}\left(\frac{\sqrt{f} - x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}, \text{ seu}$$

$$y = \frac{1}{2\sqrt{b}}\left[\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right]$$

$$+\frac{nx}{2m\sqrt{f}}\left[\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} + a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right],$$

vel posito $y = A + Bx$, erit $B = \frac{n\sqrt{(AAb+a)}}{m\sqrt{f}}$, ita vt constans B definiatur ex constante

A =

$$A = \frac{1}{2\sqrt{b}}\left[\left(\frac{\sqrt{k}}{\sqrt{b}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - a\left(\frac{\sqrt{b}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right]:$$

et viciſſim

$$\left(\frac{\sqrt{k}}{\sqrt{b}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} = A\sqrt{b} + \sqrt{(a + b\,A\,A)}, \text{ atque}$$

$$a\left(\frac{\sqrt{b}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} = -A\sqrt{b} + \sqrt{(a + b\,A\,A)}.$$

Nunc ad ſeriem inueniendam, aequatio propoſita, ſumtis quadratis

$$mm\,(f + g\,x\,x)\,\partial y^2 = nn\,(a + b\,y\,y)\,\partial x^2,$$

denuo differentietur, capto ∂x conſtante, vt facta diuiſione per $2\,\partial y$ prodeat:

$$mm\,\partial\partial y\,(f + g\,x\,x) + mm\,g\,x\,\partial x\,\partial y - nn\,b\,y\,\partial x^2 = 0.$$

Iam pro y fingatur ſeries:

$$y = A + B\,x + C\,x^2 + D\,x^3 + E\,x^4 + F\,x^5 + \text{etc.}$$

qua ſubſtituta habebitur:

$$\left.\begin{array}{l} 2mmfC + 6mmfDx + 12mmfEx^2 + 20mmfF\,x^3 + \text{etc.} \\ \qquad\qquad + 2mmgC \;\; + 6mmgD \;\; + \text{etc.} \\ \quad + mmgb\,B + 2mmgC \;\; + 3mmgD \;\; + \text{etc.} \\ - nnbA - nnb\;B - nnb\;C \;\; - nnb\;D \;\; - \text{etc.} \end{array}\right\} = 0.$$

Cum ergo A et B dentur, reliquae litterae ita determinantur:

$$C = \frac{nnb}{2mmf}\,A;$$

$$D = \frac{nnb - mmg}{2.3\,mmf}\,B;\quad E = \frac{nnb - 4mmg}{3.4\,mmf}\,C;$$

$$E = \frac{nnb - 9mmg}{4.5\,mmf}\,D;\quad G = \frac{nnb - 15mmg}{5.6\,mmf}\,E;$$

$$H = \frac{nnb - 25mmg}{6.\;mmf}\,F;\quad J = \frac{nnb - 36mmg}{7.8\,mmf}\,G;$$

ſicque ſeries pro y erit cognita.

Ex-

Exemplum 1.

186. *Functionem transcendentem* $c^{\text{Arc. fin. } x}$ *per feriem fecundum poteftates ipfius* x *progredientem exprimere.*

Ponatur $y = c^{\text{Arc. fin. } x}$, erit $l y = l c$. Arc. fin. x, et $\frac{\partial y}{y} = \frac{\partial x\, l c}{\sqrt{(1 - x x)}}$: hinc $\partial y^2 (1 - x x) = y y\, \partial x^2 (l c)^2$, et differentiando $\partial \partial y (1 - x x) - x\, \partial x\, \partial y - y\, \partial x^2 (l c)^2 = 0$. Obferuetur iam, pofito x euanefcente, fore $y = c^x = 1 + x\, l c$; hinc fingatur feries $y = 1 + x\, l c + A x^2 + B x^3 + C x^4 + D x^5 +$ etc. qua fubftituta habebitur:

$$\left.\begin{array}{llllll} 1.2\,A & + 2.3\,B x & + 3.4\,C x^2 & + 4.5\,D x^3 & + 5.6\,E x^4 & \\ & & - 1.2\,A & - 2.3\,B & - 3.4\,C & \\ & - l c & - 2\,A & - 3\,B & - 4\,C & \text{etc.} \\ - (l c)^2 & - (l c)^3 & - A (l c)^2 & - B (l c)^2 & - C (l c)^2 & \end{array}\right\} = 0.$$

Vnde reliqui coefficientes ita definiuntur:

$$A = \frac{(l c)^2}{1.2}; \qquad B = \frac{[1 + (l c)^2]\, l c}{2.3};$$

$$C = \frac{4 + (l c)^2}{3.4} A; \quad D = \frac{9 + (l c)^2}{4.5} B;$$

$$E = \frac{16 + (l c)^2}{5.6} C; \quad F = \frac{25 + (l c)^2}{6.7} D; \text{ etc.}$$

Sit breuitatis gratia $l c = \gamma$, eritque

$$c^{\text{Arc. fin. } x} = 1 + \gamma x + \frac{\gamma\gamma}{1.2} x^2 + \frac{\gamma(1 + \gamma\gamma)}{1.2.3} x^3 + \frac{\gamma\gamma(4 + \gamma\gamma)}{1.2.3.4} x^4$$
$$+ \frac{\gamma(1 + \gamma\gamma)(9 + \gamma\gamma)}{1.2.3.4.5} x^5 + \frac{\gamma\gamma(4 + \gamma\gamma)(16 + \gamma\gamma)}{1.2.3.4.5.6} x^6 + \text{etc.}$$

Exemplum 2.

187. *Pofito* $x = \text{fin.}\,\Phi$, *inuenire feriem fecundum poteftates ipfius* x *progredientem, quae finum anguli* $n\Phi$ *exprimat.*

Ponatur $y = \text{fin.}\, n\Phi$, ac notetur euanefcente Φ, fieri $x = \Phi$ et $y = n\Phi = n x$, hoc eft $y = 0 + n x$, quod eft feriei

O quae-

quaesitae initium. Nunc autem est

$\partial \Phi = \frac{\partial x}{\sqrt{(1-xx)}}$, et $n \partial \Phi = \frac{\partial y}{\sqrt{(1-yy)}}$. Ergo

$\frac{\partial y}{\sqrt{(1-yy)}} = \frac{n \partial x}{\sqrt{(1-xx)}}$,

et sumtis quadratis

$$(1-xx)\partial y^2 = nn \partial x^2 (1-yy): \text{ hinc}$$

$$\partial\partial y (1-xx) - x \partial x \partial y + nny \partial x^2 = 0.$$

Quare fingatur haec series

$$y = nx + Ax^3 + Bx^5 + Cx^7 + Dx^9 + \text{etc.}$$

qua substituta habebitur:

$$\left.\begin{array}{llll} 2.3Ax & +4.5Bx^3 & +6.7Cx^5 & +8.9Dx^7 \\ & -2.3A & -4.5B & -6.7C \\ -n & -\;\;3A & -\;\;5B & -\;\;7C \\ +n^3 & +nnA & +nnB & +nnC \end{array}\text{etc.}\right\} = 0.$$

Vnde hae determinationes colliguntur:

$$A = \frac{-n(nn-1)}{2.3};\; B = \frac{-(nn-9)A}{4.5};\; C = \frac{-(nn-25)B}{6.7} \text{ etc.}$$

ita vt sit:

$$y = nx - \frac{n(nn-1)}{1.2.3}x^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5 - \frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7 + \text{etc.}$$

siue

$$\text{sin.}\, n\Phi = n \,\text{sin.}\,\Phi - \frac{n(nn-1)}{1.2.3}\text{sin.}\,\Phi^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}\text{sin.}\,\Phi^5 - \text{etc.}$$

Scholion.

188. Quia haec series tantum casibus, quibus n est numerus impar abrumpitur, pro paribus notandum est, seriem commode exprimi posse per productum ex sin. Φ in aliam seriem, secundum cosinus ipsius Φ potestates progredientem. Ad quam inueniendam ponamus cos. $\Phi = u$, fitque sin. $n\Phi = z$ sin. $\Phi = z\sqrt{(1-uu)}$; vnde ob $\partial\Phi = -\frac{\partial u}{\sqrt{(1-uu)}}$, erit differentiando

$-n\partial u$

$$-\frac{n\,\partial u\,cos.\,n\,\Phi}{\sqrt{(1-uu)}} = \partial z\sqrt{(1-uu)} - \frac{zu\,\partial u}{\sqrt{(1-uu)}},\ \text{seu}$$

$$-n\,\partial u\,\text{cos}.\,n\,\Phi = \partial z\,(1-uu) - zu\,\partial u,$$

quae sumto ∂u constante denuo differentiata dat: $-\frac{nn\,\partial u^2\,sin.\,n\,\Phi}{\sqrt{(1-uu)}}$ $= \partial\partial z\,(1-uu) - 3u\,\partial u\,\partial z - z\,\partial u^2 = -nnz\,\partial u^2$, ob $\frac{sin.\,n\,\Phi}{\sqrt{(1-uu)}} = z$. Quocirca series quaesita pro $z = \frac{sin.\,n\,\Phi}{sin.\,\Phi}$ ex hac aequatione erui debet:

$$\partial\partial z\,(1-uu) - 3u\,\partial u\,\partial z - z\,\partial u^2 + nnz\,\partial u^2 = 0:$$

vbi notandum est, quia $u = \text{cos}.\,\Phi$ euanescente u, quo casu fit $\Phi = 90^\circ$, fore vel $z = 0$, si n numerus par, vel $z = 1$, si $n = 4\alpha + 1$; vel $z = -1$, si $n = 4\alpha - 1$. Qui singuli casus seorsim sunt euoluendi: et quo principium cuiusque seriei pateat, sit $\Phi = 90^\circ - \omega$, et euanescente ω, fit $u = \text{cos}.\,\Phi = \omega$; $\text{sin}.\,\Phi = 1$; $\text{sin}.\,n\,\Phi = \text{sin}.\,(90.\,n - n\,\omega) = z$.

Nunc pro casibus singulis:

I. si $n = 4\alpha$; fit $z = -\text{sin}.\,n\,\omega = -nu$

II. si $n = 4\alpha + 1$; fit $z = \text{cos}.\,n\,\omega = 1$

III. si $n = 4\alpha + 2$; fit $z = \text{sin}.\,n\,\omega = +nu$

IV. si $n = 4\alpha + 3$; fit $z = -\text{cos}.\,n\,\omega = -1$

vnde series iam satis notae deducuntur.

CAPVT IV.

DE
INTEGRATIONE FORMVLARVM LOGARITHMICARVM ET EXPONENTIALIVM.

Problema 18.

189.

Si X defignet functionem algebraicam ipfius x, inuenire integrale formulae $X \partial x \, l x$.

Solutio.

Quaeratur integrale $\int X \partial x$, quod fit $= Z$, et cum quantitatis $Z \, l x$ differentiale fit $= \partial Z \, l x + \frac{Z \partial x}{x}$, erit $Z \, l x = \int \partial Z \, l x + \int \frac{Z \partial x}{x}$: ideoque

$$\int \partial Z \, l x = \int X \partial x \, l x = Z \, l x - \int \frac{Z \partial x}{x}.$$

Sicque integratio formulae propofitae reducta eft ad integrationem huius $\frac{Z \partial x}{x}$, quae, fi Z fuerit functio algebraica ipfius x, non amplius logarithmum inuoluit, ideoque per praecedentes regulas tractari poterit. Sin autem $\int X \partial x$ algebraice exhiberi nequeat, hinc nihil fubfidii nafcitur, expedietque indicatione integralis $\int X \partial x \, l x$ acquiefcere, eiusque valorem per approximationem inueftigare.

Nifi forte fit $X = \frac{1}{x}$, quo cafu manifefto dat $\int \frac{\partial x}{x} l x = \frac{1}{2} (l x)^2 + C$.

Corollarium 1.

190. Eodem modo, fi denotante V functionem quamcunque ipfius x, propofita fit formula $X \partial x \, l V$, erit exiftente $\int X \partial x = Z$, eius integrale $= Z \, l V - \int \frac{Z \partial V}{V}$, ficque ad formulam algebraicam reducitur, fi modo Z algebraice detur.

Co-

Corollarium 2.

191. Pro casu singulari $\frac{\partial x}{x} l x$ notare licet, si posito $l x = u$, fuerit U functio quaecunque algebraica ipsius u, integrationem huius formulae $\frac{U \partial x}{x}$ non fore difficilem, quia ob $\frac{\partial x}{x} = \partial u$ abit in $U \partial u$, cuius integratio ad praecedentia capita refertur.

Scholion.

192. Haec reductio innititur isti fundamento, quod cum sit $\partial . x y = y \partial x + x \partial y$, hinc vicissim fiat $x y = \int y \partial x + \int x \partial y$, ita vt hoc modo in genere integratio formulae $y \partial x$ ad integrationem formulae $x \partial y$ reducatur. Quod si ergo, proposita quacunque formula $V \partial x$, functio V in duos factores, puta $V = P Q$, resolui queat, ita vt integrale $\int P \partial x = S$ assignari queat, ob $P \partial x = \partial S$, erit $V \partial x = P Q \partial x = Q \partial S$, hincque $\int V \partial x = Q S - \int S \partial Q$. Huiusmodi reductio insignem vsum affert, cum formula $\int S \partial Q$ simplicior fuerit quam proposita $\int V \partial x$, eaque insuper simili modo ad simpliciorem reduci queat. Interdum etiam commode euenit, vt hac methodo tandem ad formulam propositae similem perueniatur, quo casu integratio pariter obtinetur. Veluti si vlteriori reductione inueniremus $\int S \partial Q = T + n \int V \partial x$, foret vtique $\int V \partial x = Q S - T - n \int V \partial x$, hincque $\int V \partial x = \frac{Q S - T}{n + 1}$. Tum igitur talis reductio insignem praestat vsum, cum vel ad formulam simpliciorem, vel ad eandem perducit. Atque ex hoc principio praecipuos casus, quibus formula $X \partial x l x$ vel integrationem admittit, vel per seriem commode exhiberi potest, euoluamus.

Exemplum 1.

193. *Formulae differentialis $x^n \partial x l x$ integrale inuenire, denotante n numerum quemcunque.*

Cum sit $\int x^n \partial x = \frac{1}{n+1} x^{n+1}$, erit

$$\int x^n \partial x \, l x = \frac{1}{n+1} x^{n+1} l x - \int \frac{1}{n+1} x^{n+1} \partial . l x$$
$$= \frac{1}{n+1} x^{n+1} l x - \frac{1}{n+1} \int x^n \partial x = \frac{1}{n+1} x^{n+1} l x$$
$$- \frac{1}{(n+1)^2} x^{n+1}; \text{ ideoque}$$
$$\int x^n \partial x \, l x = \frac{1}{n+1} x^{n+1} \left(l x - \frac{1}{n+1}\right).$$

Sicque haec formula absolute est integrabilis.

Corollarium 1.

194. Casus simpliciores, quibus n est numerus integer siue positiuus siue negatiuus, tenuisse iuuabit:

$$\int \partial x \, l x = x \, l x - x; \qquad \int \frac{\partial x}{xx} l x = -\frac{1}{x} l x - \frac{1}{x};$$
$$\int x \partial x \, l x = \tfrac{1}{2} x x \, l x - \tfrac{1}{4} x x; \quad \int \frac{\partial x}{x^3} l x = -\frac{1}{2 x x} l x - \frac{1}{4 x x};$$
$$\int x^2 \partial x \, l x = \tfrac{1}{3} x^3 l x - \tfrac{1}{9} x^3; \quad \int \frac{\partial x}{x^4} l x = -\frac{1}{3 x^3} l x - \frac{1}{9 x^3};$$
$$\int x^3 \partial x \, l x = \tfrac{1}{4} x^4 l x - \tfrac{1}{16} x^4; \quad \int \frac{\partial x}{x^5} l x = -\frac{1}{4 x^4} l x - \frac{1}{16 x^4}.$$

Corollarium 2.

195. Casum $\int \frac{\partial x}{x} l x = \frac{1}{2} (l x)^2$, qui est omnino singularis, iam supra annotauimus, sequitur vero etiam ex reductione ad eandem formulam. Namque per superiorem reductionem habemus

$$\int \frac{\partial x}{x} l x = l x . l x - \int l x . \partial l x = (l x)^2 - \int \frac{\partial x}{x} l x:$$

hincque

$$2 \int \frac{\partial x}{x} l x = (l x)^2, \text{ consequenter } \int \frac{\partial x}{x} l x = \tfrac{1}{2} (l x)^2.$$

Exemplum 2.

196. *Formulae* $\frac{\partial x}{1-x} l x$ *integrale per seriem exprimere.*

Reductione ante adhibita parum lucramur, prodit enim:

$$\int \frac{\partial x}{1-x} l x = l \frac{1}{1-x} . l x - \int \frac{\partial x}{x} l \frac{1}{1-x}.$$

Cum

Cum autem sit

$$l\frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \text{etc. erit}$$

$$\int\frac{\partial x}{x}\,l\frac{1}{1-x} = x + \tfrac{1}{4}x^2 + \tfrac{1}{9}x^3 + \tfrac{1}{16}x^4 + \tfrac{1}{25}x^5 + \text{etc.}$$

ideoque

$$\int\frac{\partial x}{1-x}\,l\,x = l\frac{1}{1-x}\,.\,l\,x - x - \tfrac{1}{4}x^2 - \tfrac{1}{9}x^3 - \tfrac{1}{16}x^4 - \tfrac{1}{25}x^5 - \text{etc.}$$

quod integrale euanescit casu $x = 0$, etsi enim $l\,x$ tum in infinitum abit, tamen $l\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3$ etc. ita euanescit, vt etiam si per $l\,x$ multiplicetur, in nihilum abeat, est enim in genere $x^n\,l\,x = 0$ posito $x = 0$, dum n numerus positiuus.

Corollarium 1.

197. Si ponamus $1 - x = u$, fit

$$\frac{\partial x}{1-x}\,l\,x = -\frac{\partial u}{u}\,l(1-u) = \frac{\partial u}{u}\,l\frac{1}{1-u},$$

ideoque

$$\int\frac{\partial x}{1-x}\,l\,x = C + u + \tfrac{1}{4}u^2 + \tfrac{1}{9}u^3 + \tfrac{1}{16}u^4 + \tfrac{1}{25}u^5 + \text{etc.}$$

quae, vt etiam casu $x = 0$ seu $u = 1$, euanescat, capi debet

$$C = -1 - \tfrac{1}{4} - \tfrac{1}{9} - \tfrac{1}{16} - \tfrac{1}{25} - \text{etc.} = -\tfrac{1}{6}\pi\pi.$$

Corollarium 2.

198. Sumto ergo $1 - x = u$ seu $x + u = 1$, aequales erunt inter se hae expressiones:

$$-l\,x\,.\,l\,u - x - \tfrac{1}{4}x^2 - \tfrac{1}{9}x^3 - \tfrac{1}{16}x^4 - \text{etc.}$$

$$= -\tfrac{1}{6}\pi^2 + u + \tfrac{1}{4}u^2 + \tfrac{1}{9}u^3 + \text{etc.}$$

seu erit

$$\tfrac{1}{6}\pi^2 - l\,x\,.\,l\,u = x + u + \tfrac{1}{4}(x^2+u^2) + \tfrac{1}{9}(x^3+u^3)$$
$$+ \tfrac{1}{16}(x^4+u^4) + \text{etc.}$$

Corollarium 3.

199. Haec series maxime conuergit, ponendo $x = u = \frac{1}{2}$: hoc ergo casu habebimus

$\frac{1}{6}\pi -$

$$\tfrac{1}{6}\pi - (l\,2)^2 = 1 + \tfrac{1}{2.4} + \tfrac{1}{4.9} + \tfrac{1}{8.16} + \tfrac{1}{16.25} + \tfrac{1}{32.36} + \text{etc.}$$

Huius ergo feriei

$$x + \tfrac{1}{4}x^2 + \tfrac{1}{9}x^3 + \tfrac{1}{16}x^4 + \tfrac{1}{25}x^5 + \text{etc.}$$

fumma habetur non folum cafu $x = 1$, quo eft $= \frac{\pi\pi}{6}$, fed etiam cafu $x = \frac{1}{2}$, quo eft $= \frac{1}{12}\pi^2 - \frac{1}{2}(l\,2)^2$.

Corollarium 4.

200. Si ponamus $x = \frac{1}{3}$, et $u = \frac{2}{3}$, erit huius feriei

$$1 + \tfrac{5}{3^2.4} + \tfrac{9}{3^3.9} + \tfrac{17}{3^4.16} + \tfrac{33}{3^5.25} + \tfrac{65}{3^6.36} + \text{etc.}$$

cuius terminus generalis $= \dfrac{1 + 2^n}{3^n\,n\,n}$, fumma $= \frac{1}{6}\pi^2 - l\,3\,.\,l\,\frac{3}{2}$: neque vero hinc feriei $x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 + \text{etc.}$ binos cafus $x = \frac{1}{3}$ et $x = \frac{2}{3}$ feorfim fummare licet.

Exemplum 3.

201. Formulae $\frac{\partial x}{(1-x)^2}\,l\,x$ *integrale inuenire, idemque in feriem conuertere.*

Cum fit $\int \frac{\partial x}{(1-x)^2} = \frac{1}{1-x}$, erit

$\int \frac{\partial x}{(1-x)^2}\,l\,x = \frac{1}{1-x}\,l\,x - \int \frac{\partial x}{x(1-x)}$: at ob

$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$, fit $\int \frac{\partial x}{x(1-x)} = l\,x + l\,\frac{1}{1-x}$,

vnde colligimus integrale

$$\int \frac{\partial x}{(1-x)^2}\,l\,x = \frac{l\,x}{1-x} - l\,x - l\,\frac{1}{1-x} = \frac{x\,l\,x}{1-x} - l\,\frac{1}{1-x},$$

ita fumtum, vt euanefcat pofito $x = 0$.

Iam pro ferie commodiffime inuenienda, ftatuatur $1 - x = u$, et noftra formula fit

$$= \frac{-\partial u}{u\,u}\,l(1-u) = \frac{\partial u}{u\,u}\,l\,\frac{1}{1-u} = \frac{\partial u}{u\,u}\left(u + \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3 + \tfrac{1}{4}u^4 + \tfrac{1}{5}u^5 + \text{etc.}\right)$$

Quocirca integrando nancifcimur:

$$\int \frac{\partial x}{(1-x)^2}\,l\,x = C + l\,u + \frac{u}{1.2} + \frac{u\,u}{2.3} + \frac{u^3}{3.4} + \frac{u^4}{4.5} + \text{etc.}$$

quae

quae expressio vt etiam euanescat, facto $x = 0$ seu $u = 1$, oportet sit:

$$C = -\frac{1}{1.2} - \frac{1}{2.3} - \frac{1}{3.4} - \frac{1}{4.5} - \text{etc.} = -1.$$

Quare ob $x = 1 - u$, obtinebimus:

$$\frac{u}{1.2} + \frac{u^2}{2.3} + \frac{u^3}{3.4} + \frac{u^4}{4.5} + \text{etc.} = 1 - lu + \frac{(1-u)l(1-u)}{u} + lu$$
$$= 1 + \frac{(1-u)l(1-u)}{u}.$$

Corollarium 1.

202. Simili modo si $\partial y = \frac{\partial u}{u\sqrt{u}} l \frac{1}{1-u}$, erit

$$y = -\frac{2}{\sqrt{u}} l \frac{1}{1-u} + \int \frac{2\partial u}{(1-u)\sqrt{u}};$$

at posito $u = xx$, fit

$$\int \frac{2\partial u}{(1-u)\sqrt{u}} = 4\int \frac{\partial x}{1-xx} = 2 l \frac{1+x}{1-x}. \quad \text{Ergo}$$
$$y = 2 l \frac{1+\sqrt{u}}{1-\sqrt{u}} - \frac{2}{\sqrt{u}} l \frac{1}{1-u}.$$

At quia pèr seriem

$$\partial y = \frac{\partial u}{u\sqrt{u}} (u + \tfrac{1}{2}uu + \tfrac{1}{3}u^3 + \tfrac{1}{4}u^4 + \text{etc.})$$

erit etiam

$$y = +2\sqrt{u} + \frac{2}{2.3}u\sqrt{u} + \frac{2}{3.5}u^2\sqrt{u} + \frac{2}{4.7}u^3\sqrt{u} + \text{etc.}$$

Corollarium 2.

203. Si ergo multiplicemus per $\frac{\sqrt{u}}{2}$, adipiscimur:

$$u + \frac{uu}{2.3} + \frac{u^3}{3.5} + \frac{u^4}{4.7} + \frac{u^5}{5.9} + \text{etc.} = \sqrt{u}.l\frac{1+\sqrt{u}}{1-\sqrt{u}} + l(1-u),$$

quae summa est etiam

$$= (1+\sqrt{u})l(1+\sqrt{u}) + (1-\sqrt{u})l(1-\sqrt{u}).$$

Quare sumto $u = 1$, ob $(1-\sqrt{u})l(1-\sqrt{u}) = 0$, erit

$$1 + \frac{1}{2.3} + \frac{1}{3.5} + \frac{1}{4.7} + \frac{1}{5.9} + \frac{1}{6.11}\text{etc.} = 2l2.$$

Problema 19.

204. Si P denotet functionem ipsius x, inuenire integrale huius formulae $\partial y = \partial P(lx)^n$.

Solutio.

Per reductionem fupra monftratam fit

$$y = P\,(l\,x)^n - \int P\,\partial.\,(l\,x)^n = P\,(l\,x)^n - n \int \tfrac{P\,\partial x}{x}\,(l\,x)^{n-1}.$$

Hinc fi fit $\int \frac{P\,\partial x}{x} = Q$, erit fimili modo

$$\int \tfrac{P\,\partial x}{x}\,(l\,x)^{n-1} = Q\,(l\,x)^{n-1} - (n-1) \int \tfrac{Q\,\partial x}{x}\,(l\,x)^{n-2}.$$

Quo modo fi vlterius progredimur, haecque integralia capere liceat

$$\int \tfrac{P\,\partial x}{x} = Q;\ \int \tfrac{Q\,\partial x}{x} = R;\ \int \tfrac{R\,\partial x}{x} = S;\ \int \tfrac{S\,\partial x}{x} = T;\ \text{etc.}$$

obtinebimus integrale quaefitum:

$$\int \partial P\,(l\,x)^n = P\,(l\,x)^n - n\,Q\,(l\,x)^{n-1} + n\,(n-1)\,R\,(l\,x)^{n-2}$$
$$- n\,(n-1)\,(n-2)\,S\,(l\,x)^{n-3} + \text{etc.}$$

ac fi exponens n fuerit numerus integer pofitiuus, integrale forma finita exprimetur.

Exemplum I.

205. *Formulae $x^m\,\partial x\,(l\,x)^2$ integrale affignare.*

Hic eft $n = 2$, et $P = \frac{x^{m+1}}{m+1}$; hinc $Q = \frac{x^{m+1}}{(m+1)^2}$, et $R = \frac{x^{m+1}}{(m+1)^3}$: vnde colligimus

$$\int x^m\,\partial x\,(l\,x)^2 = x^{m+1}\left(\tfrac{(l\,x)^2}{m+1} - \tfrac{2\,l\,x}{(m+1)^2} + \tfrac{2\,.\,1}{(m+1)^3}\right),$$

quod integrale euanefcit pofito $x = 0$, dum fit $m + 1 > 0$.

Corollarium I.

206. Hinc pofito $x = 1$, fit $\int x^m\,\partial x\,(l\,x)^2 = \frac{2\,.\,1}{(m+1)^3}$. Ex praecedentibus autem patet, fi formula $\int x^m\,\partial x\,l\,x$ ita integretur, vt euanefcat pofito $x = 0$, tum facto $x = 1$, fieri $\int x^m\,\partial x\,l\,x = \frac{-1}{(m+1)^2}$.

Corol-

Corollarium 2.

207. At si sit $m = -1$, vt habeatur $\frac{\partial x}{x}(l\,x)^2$, erit eius integrale $\int \frac{\partial x}{x}(l\,x)^2 = \frac{1}{3}(l\,x)^3$, qui solus casus ex formula generali est excipiendus.

Exemplum 2.

208. *Formulae* $x^{m-1}\partial x\,(l\,x)^3$ *integrale assignare.*

Hic est $n = 3$ et $P = \frac{x^m}{m}$, hinc $Q = \frac{x^m}{m^2}$; $R = \frac{x^m}{m^3}$ et $S = \frac{x^m}{m^4}$: vnde integrale quaesitum fit

$$\int x^{m-1}\partial x\,(l\,x)^3 = x^m\left(\frac{(l\,x)^3}{m} - \frac{3(l\,x)^2}{m^2} + \frac{3.2\,l\,x}{m^3} - \frac{3.2.1}{m^4}\right),$$

quod integrale euanescit, posito $x = 0$, dum sit $m > 0$.

Corollarium 1.

209. Quod si integrali ita sumto, vt euanescat posito $x = 0$, tum ponatur $x = 1$, erit:

$\int x^{m-1}\partial x = \frac{1}{m}$; $\int x^{m-1}\partial x\,l\,x = -\frac{1}{m^2}$; $\int x^{m-1}\partial x\,(l\,x)^2 = +\frac{1.2}{m^3}$; et $\int x^{m-1}\partial x\,(l\,x)^3 = -\frac{1.2.3}{m^4}$.

Corollarium 2.

210. Casu autem $m = 0$, erit integrale

$$\int \frac{\partial x}{x}(l\,x)^3 = \frac{1}{4}(l\,x)^4,$$

quod ita determinari nequit, vt euanescat posito $x = 0$; oportet enim constantem infinitam adiici. Hoc autem integrale euanescit posito $x = 1$.

Exemplum 3.

211. *Formulae $x^{m-1}\partial x\,(l\,x)^n$ integrale assignare.*

Cum hic sit $P=\frac{x^m}{m}$; erit $Q=\frac{x^m}{m^2}$; $R=\frac{x^m}{m^3}$; $S=\frac{x^m}{m^4}$; etc. Hinc integrale quaesitum prodit

$$\int x^{m-1}\partial x\,(l\,x)^n = x^m\left(\frac{(l\,x)^n}{m}-\frac{n\,(l\,x)^{n-1}}{m^2}+\frac{n\,(n-1)\,(l\,x)^{n-2}}{m^3}\right.$$
$$\left.-\frac{n\,(n-1)\,(n-2)\,(l\,x)^{n-3}}{m^4}+\text{etc.}\right).$$

Casu autem $m=0$, est $\int\frac{\partial x}{x}(l\,x)^n=\frac{1}{n+1}(l\,x)^{n+1}$.

Corollarium 1.

212. Si $m>0$ integrale assignatum euanescit, posito $x=0$: deinceps ergo si sumatur $x=1$, erit integrale

$$\int x^{m-1}\partial x\,(l\,x)^n=\pm\frac{1.\,2.\,3.\,\ldots\ldots\,n}{m^{n+1}},$$

vbi signum $+$ valet, si n sit numerus par, inferius vero $-$ si n impar.

Corollarium 2.

213. Haec ergo ambiguitas tollitur, si loco $l\,x$ scribatur $-l\frac{1}{x}$; tum enim integratione eodem modo instituta, positoque $x=1$, fiet

$$\int x^{m-1}\partial x\,(l\tfrac{1}{x})^n=+\frac{1.\,2.\,3.\,\ldots\ldots\,n}{m^{n+1}}.$$

Scholion.

214. Si exponens n sit numerus fractus, integrale inventum per seriem infinitam exprimitur, veluti si sit $n=-\frac{1}{2}$, reperitur

$\int x^m$

$$\int \frac{x^{m-1}\,\partial x}{\sqrt{l\,x}} = x^m \left(\frac{1}{m\sqrt{l\,x}} + \frac{1}{2\,m^2 (l\,x)^{\frac{3}{2}}} + \frac{1.\,3}{4\,m^3 (l\,x)^{\frac{5}{2}}} + \frac{1.\,3.\,5}{8\,m^4 (l\,x)^{\frac{7}{2}}} + \text{etc.} \right),$$

quae etiam quatenus initio x ab 0 ad 1 crefcere fumitur, hoc modo repraefentari poteft:

$$\int \frac{x^{m-1}\,\partial x}{\sqrt{l\,\frac{1}{x}}} = \frac{x^m}{\sqrt{l\,\frac{1}{x}}} \left(\frac{1}{m} + \frac{1}{2\,m^2\,l\,x} + \frac{1.\,3}{4\,m^3 (l\,x)^2} + \frac{1.\,3.\,5}{8\,m^4 (l\,x)^3} + \text{etc.} \right).$$

Si exponens n fit negatiuus, etfi integer, tamen integrale inventum in infinitum progreditur: verum hoc cafu alia ratione integrationem inftituere licet, qua tandem reducitur ad huiusmodi formulam $\int \frac{\mathrm{T}\,\partial x}{l\,x}$, cuius integratio nullo modo fimplicior reddi poteft. Hanc ergo reductionem fequenti problemate doceamus.

Problema 20.

215. Integrationem huius formulae $\partial y = \frac{\mathrm{X}\,\partial x}{(l\,x)^n}$ continuo ad formulas fimpliciores reducere.

Solutio.

Formula propofita ita repraefentetur $\partial y = \mathrm{X}x \,.\, \frac{\partial x}{x\,(l\,x)^n}$, et cum fit $\int \frac{\partial x}{x\,(l\,x)^n} = \frac{-1}{(n-1)\,(l\,x)^{n-1}}$, erit

$$y = \frac{-\mathrm{X}\,x}{(n-1)\,(l\,x)^{n-1}} + \frac{1}{n-1} \int \frac{1}{(l\,x)^{n-1}} \,.\, \partial\,(\mathrm{X}\,x).$$

 Quare

Quare si ponamus continuo

$$\partial.(\mathrm{X}x) = \mathrm{P}\,\partial x;\ \partial.(\mathrm{P}x) = \mathrm{Q}\,\partial x;\ \partial.(\mathrm{Q}x) = \mathrm{R}\,\partial x \text{ etc.}$$

erit hanc reductionem continuando:

$$y = \frac{-\mathrm{X}x}{(n-1)(lx)^{n-1}} - \frac{\mathrm{P}x}{(n-1)(n-2)(lx)^{n-2}} - \frac{\mathrm{Q}x}{(n-1)(n-2)(n-3)(lx)^{n-3}} - \text{etc.}$$

donec tandem perueniatur ad hanc integralem

$$+ \frac{1}{(n-1)(n-2)\ldots 1}\int\frac{\mathrm{V}\,\partial x}{lx},$$

ita vt quoties n fuerit numerus integer positiuus, integratio tandem ad huiusmodi formulam perducatur.

Exemplum 1.

216. *Formulae differentialis* $\partial y = \frac{x^{m-1}\,\partial x}{(lx)^2}$ *integrale inuestigare.*

Hic est $n = 2$ et $\mathrm{X} = x^{m-1}$, vnde fit $\mathrm{P} = m\,x^{m-1}$, hincque integrale

$$y = \int\frac{x^{m-1}\,\partial x}{(lx)^2} = -\frac{x^m}{lx} + \frac{m}{1}\int\frac{x^{m-1}\,\partial x}{lx}.$$

At formulae $\frac{x^{m-1}\,\partial x}{lx}$ integrale exhiberi nequit, nisi casu $m = 0$, quo fit $\int\frac{\partial x}{x\,lx} = l\,l\,x$. Verum si $m = 0$, formulae propositae integratio ne hinc quidem pendet: fit enim absolute $y = \int\frac{\partial x}{x(lx)^2} = -\frac{1}{lx} + \mathrm{C}$.

Exemplum 2.

217. *Formulae differentialis* $\partial y = \frac{x^{m-1}\,\partial x}{(lx)^n}$ *integrale inuestigare casibus, quibus* n *est numerus integer positiuus.*

Cum

Cum fit $X = x^{m-1}$, erit $P = \frac{\partial . (X x)}{\partial x} = m x^{m-1}$, tum vero $Q = \frac{\partial P x}{\partial x} = m^2 x^{m-1}$; $R = m^3 x^{m-1}$; $S = m^4 x^{m-1}$; etc. Quare integrale hinc ita formabitur, vt fit

$$y = \int \frac{x^{m-1} \partial x}{(l x)^n} = \frac{-x^m}{(n-1)(l x)^{n-1}} - \frac{m x^m}{(n-1)(n-2)(l x)^{n-2}}$$

$$- \frac{m^2 x^m}{(n-1)(n-2)(n-3)(l x)^{n-3}} - \text{etc.}$$

$$\ldots + \frac{m^{n-1}}{(n-1)(n-2) \ldots \ldots 1} \int \frac{x^{m-1} \partial x}{l x}.$$

Corollarium.

218. Pro n ergo fucceffiue numeros 1, 2, 3, 4, etc. fubftituendo, habebimus iftas reductiones:

$$\int \frac{x^{m-1} \partial x}{(l x)^2} = \frac{-x^m}{l x} + \frac{m}{1} \int \frac{x^{m-1} \partial x}{l x}$$

$$\int \frac{x^{m-1} \partial x}{(l x)^3} = \frac{-x^m}{2 (l x)^2} - \frac{m x^m}{2 . 1 \, l x} + \frac{m^2}{2 . 1} \int \frac{x^{m-1} \partial x}{l x}$$

$$\int \frac{x^{m-1} \partial x}{(l x)^4} = \frac{-x^m}{3 (l x)^3} - \frac{m x^m}{3 . 2 (l x)^2} - \frac{m^2 x^m}{3 . 2 . 1 \, l x} + \frac{m^3}{3 . 2 . 1} \int \frac{x^{m-1} \partial x}{l x}.$$

Scholion.

219. Hae ergo integrationes pendent a formula $\int \frac{x^{m-1} \partial x}{l x}$, quae pofito $x^m = z$, ob $x^{m-1} \partial x = \frac{1}{m} \partial z$ et $l x = \frac{1}{m} l z$, reducitur ad hanc fimpliciffimam formam $\int \frac{\partial z}{l z}$, cuius integrale fi affignari poffet, ampliffimum vfum in Analyfi effet allaturum, verum nullis adhuc artificiis, neque per logarithmos, neque angulos, exhiberi potuit: quomodo autem per feriem exprimi poffit, infra oftendemus (§. 227.). Videtur ergo

ergo haec formula $\int\frac{\partial z}{lz}$ singularem speciem functionum transcendentium suppeditare, quae vtique accuratiorem euolutionem meretur. Eadem autem quantitas transcendens in integrationibus formularum exponentialium frequenter occurrit, quas in hoc capite tractare instituimus, propterea quod cum logarithmicis tam arcte cohaerent, vt alterum genus facile in alterum conuerti possit: veluti ipsa formula modo considerata $\frac{\partial z}{lz}$, posito $lz = x$, vt fit $z = e^x$, et $\partial z = e^x \partial x$, transformatur in hanc exponentialem $e^x\frac{\partial x}{x}$, cuius ergo integratio aeque est abscondita. Formulas igitur tractabiles euoluamus et eiusmodi quidem, quae non obuia substitutione ad formam algebraicam reduci possunt. Veluti si V fuerit functio quaecunque ipsius v, sitque $v = a^x$, formula $V\partial x$, ob $x = \frac{lv}{la}$ et $\partial x = \frac{\partial v}{v\,la}$, abit in $\frac{V\partial v}{v\,la}$, quae ratione variabilis v est algebraica. Huiusmodi ergo formulas $\frac{a^x \partial x}{\sqrt{(1 + a^{nx})}}$, quippe quae posito $a^x = v$, nihil habent difficultatis, hinc excludimus.

Problema 21.

220. Formulae differentialis $a^x X \partial x$, denotante X functionem quamcunque ipsius x, integrale inuestigare.

Solutio 1.

Cum sit $\partial. a^x = a^x \partial x\, la$, viciffim $\int a^x \partial x = \frac{1}{la} a^x$: quare si formula proposita in hos factores resoluatur, $X.a^x\partial x$, habebitur per reductionem:

$$\int a^x X \partial x = \frac{1}{la} a^x X - \frac{1}{la}\int a^x \partial X.$$

Quodsi vlterius ponamus $\partial X = P\partial x$, vt fit

$$\int a^x P \partial x = \frac{1}{la} a^x P - \frac{1}{la}\int a^x \partial P,$$

pro-

prodibit haec reductio

$$\int a^x X \partial x = \frac{1}{l a} a^x X - \frac{1}{(l a)^2} a^x P + \frac{1}{(l a)^2} \int a^x \partial P.$$

Si porro ponamus $\partial P = Q \partial x$, habebitur haec reductio

$$\int a^x X \partial x = \frac{1}{l a} a^x X - \frac{1}{(l a)^2} a^x P + \frac{1}{(l a)^3} a^x Q - \frac{1}{(l a)^4} \int a^x \partial Q;$$

ficque vlterius ponendo $\partial Q = R \partial x$, $\partial R = S \partial x$, etc. progredi licet, donec ad formulam vel integrabilem, vel in fuo genere fimpliciffimam perueniatur.

Solutio 2.

Alio modo refolutio formulae in factores inftitui poteft; ponatur $\int X \partial x = P$ feu $X \partial x = \partial P$, et formula ita relata $a^x . \partial P$, habebitur

$$\int a^x X \partial x = a^x P - l a \int a^x P \partial x:$$

fimili modo fi ponamus $\int P \partial x = Q$, obtinebimus

$$\int a^x X \partial x = a^x P - l a. a^x Q + (l a)^2 \int a^x Q \partial x.$$

Ponamus porro $\int Q \partial x = R$, et confequimur

$$\int a^x X \partial x = a^x P - l a. a^x Q + (l a)^2. a^x R - (l a)^3 \int a^x R \partial x,$$

hocque modo quousque lubuerit progredi licet, donec ad formulam vel integrabilem vel in fuo genere fimpliciffimam perveniamus.

Corollarium 1.

221. Priori folutione femper vti licet, quia functiones P, Q, R, etc. per differentiationem functionis X eliciuntur, dum eft

$$P = \frac{\partial X}{\partial x}; \; Q = \frac{\partial P}{\partial x}; \; R = \frac{\partial Q}{\partial x}; \text{ etc.}$$

Quare fi X fuerit functio rationalis integra, tandem ad formulam peruenietur $\int a^x \partial x = \frac{1}{l a} . a^x$; ideoque his cafibus integrale abfolute exhiberi poteft.

Q Corol-

Corollarium 2.

222. Altera folutio locum non inuenit, nifi formulae $X \partial x$ integrale P affignari queat; neque etiam eam continuare licet, nifi quatenus fequentes integrationes $\int P \partial x = Q$, $\int Q \partial x = R$, etc. fuccedunt.

Exemplum 1.

223. *Formulae $a^x x^n \partial x$ integrale definire, denotante* n *numerum integrum pofitiuum.*

Cum fit $X = x^n$, folutione prima vtentes habebimus

$$\int a^x x^n \partial x = \frac{1}{la} . a^x x^n - \frac{n}{la} \int a^x x^{n-1} \partial x;$$

hinc ponendo pro n fucceffiue numeros 0, 1, 2, 3, etc. quia primo cafu integratio conftat, fequentia integralia eruemus:

$$\int a^x \partial x = \frac{1}{la} . a^x$$

$$\int a^x x \partial x = \frac{1}{la} . a^x x - \frac{1}{(la)^2} a^x$$

$$\int a^x x^2 \partial x = \frac{1}{la} . a^x x^2 - \frac{2}{(la)^2} a^x x + \frac{2.1}{(la)^3} a^x$$

$$\int a^x x^3 \partial x = \frac{1}{la} . a^x x^3 - \frac{3}{(la)^2} a^x x^2 + \frac{3.2}{(la)^3} a^x x - \frac{3.2.1}{(la)^4} a^x$$

etc.

vnde in genere pro quouis exponente n concludimus

$$\int a^x x^n \partial x = a^x \left(\frac{x^n}{la} - \frac{n x^{n-1}}{(la)^2} + \frac{n(n-1) x^{n-2}}{(la)^3} - \frac{n(n-1)(n-2) x^{n-3}}{(la)^4} + \text{etc.} \right).$$

ad quam expreffionem info\-per conftantem arbitrariam adiici oportet, vt integrale completum obtineatur.

Corollarium.

224. Si integrale ita determinari debeat, vt euanefcat pofito $x = 0$, erit

$\int a^x$

$$\int a^x \partial x = \frac{1}{l\,a} \cdot a^x - \frac{1}{l\,a}$$
$$\int a^x x \partial x = a^x \left(\frac{x}{l\,a} - \frac{1}{(l\,a)^2}\right) + \frac{1}{(l\,a)^2}$$
$$\int a^x x^2 \partial x = a^x \left(\frac{x^2}{(l\,a)} - \frac{2x}{(l\,a)^2} + \frac{2.1}{(l\,a)^3}\right) - \frac{2.1}{(l\,a)^3}$$
$$\int a^x x^3 \partial x = a^x \left(\frac{x^3}{l\,a} - \frac{3x^2}{(l\,a)^2} + \frac{3.2x}{(l\,a)^3} - \frac{3.2.1}{(l\,a)^4}\right) + \frac{3.2.1}{(l\,a)^4}$$

etc.

Exemplum 2.

225. *Formulae* $\frac{a^x \partial x}{x^n}$ *integrale inuestigare, si quidem* n *denotet numerum integrum positiuum.*

Hic commode altera solutione vtemur, vbi cum sit $X = \frac{1}{x^n}$, erit $P = \frac{-1}{(n-1)\,x^{n-1}}$; hincque resultat ista reductio

$$\int \frac{a^x \partial x}{x^n} = \frac{-a^x}{(n-1)\,x^{n-1}} + \frac{l\,a}{n-1} \int \frac{a^x \partial x}{x^{n-1}}.$$

Perspicuum igitur est, posito $n = 1$ hinc nihil concludi posse; qui est ipse casus supra memoratus $\int \frac{a^x \partial x}{x}$, singularem speciem transcendentium functionum complectens, qua admissa integralia sequentium casuum exhibere poterimus:

$$\int \frac{a^x \partial x}{x^2} = C - \frac{a^x}{1\,x} + \frac{l\,a}{1} \int \frac{a^x \partial x}{x}$$
$$\int \frac{a^x \partial x}{x^3} = C - \frac{a^x}{2\,x^2} - \frac{a^x\,l\,a}{2.1\,x} + \frac{(l\,a)^2}{2.1} \int \frac{a^x \partial x}{x}$$
$$\int \frac{a^x \partial x}{x^4} = C - \frac{a^x}{3\,x^3} - \frac{a^x\,l\,a}{3.2\,x^2} - \frac{a^x\,(l\,a)^2}{3.2.1\,x} + \frac{(l\,a)^3}{3.2.1} \int \frac{a^x \partial x}{x}$$

vnde in genere colligimus

 $\int a^x$

$$\int \frac{a^x \partial x}{x^n} = C - \frac{a^x}{(n-1)x^{n-1}} - \frac{a^x l a}{(n-1)(n-2)x^{n-2}}$$
$$- \frac{a^x (l a)^2}{(n-1)(n-2)(n-3)x^{n-3}} - \ldots\ldots - \frac{a^x (l a)^{n-2}}{(n-1)(n-2)\ldots .1.x}$$
$$+ \frac{(l a)^{n-1}}{(n-1)(n-2)\ldots\ldots 1} \int \frac{a^x \partial x}{x}.$$

Corollarium 1.

226. Admiſſa ergo quantitate tranſcendente $\int \frac{a^x \partial x}{x}$, hanc formulam $a^x x^m \partial x$ integrare poterimus, ſiue exponens m fuerit numerus integer poſitiuus, ſiue negatiuus. Illis quidem caſibus integratio ab iſta noua quantitate tranſcendente non pendet.

Corollarium 2.

227. At ſi m fuerit fractus numerus, neutra ſolutio negotium conficit, ſed vtraque ſeriem infinitam pro integrali exhibet. Veluti ſi ſit $m = -\frac{1}{2}$, habebimus ex priore

$$\int \frac{a^x \partial x}{\sqrt{x}} = a^x \left(\frac{1}{l a} + \frac{1}{2x(l a)^2} + \frac{1.3}{4x^2(l a)^3} + \frac{1.3.5}{8x^3(l a)^4} + \text{etc.} \right) : \sqrt{x} + C,$$

ex poſteriore autem;

$$\int \frac{a^x \partial x}{\sqrt{x}} = C + \frac{a^x}{\sqrt{x}} \left(\frac{2x}{1} - \frac{4x^2 l a}{1.3} + \frac{8x^3 (l a)^2}{1.3.5} - \frac{16 x^4 (l a)^3}{1.3.5.7} + \text{etc.} \right).$$

Scho-

Scholion 1.

228. Hinc quantitas transcendens $\int \frac{a^x \partial x}{x}$ per seriem exprimi potest secundum potestates ipsius x progredientem. Cum enim sit

$$a^x = 1 + x\,l\,a + \frac{x^2 (l\,a)^2}{1.2} + \frac{x^3 (l\,a)^3}{1.2.3} + \text{etc. erit}$$

$$\int \frac{a^x \partial x}{x} = C + l\,x + \frac{x\,l\,a}{1} + \frac{x^2 (l\,a)^2}{1.2.2} + \frac{x^3 (l\,a)^3}{1.2.3.3} + \frac{x^4 (l\,a)^4}{1.2.3.4.4} + \text{etc.}$$

Ac si pro a sumamus numerum, cuius logarithmus hyperbolicus est vnitas, quem numerum littera e indicemus, habebimus

$$\int \frac{e^x \partial x}{x} = C + l\,x + \frac{x}{1} + \frac{1}{2} \cdot \frac{x^2}{1.2} + \frac{1}{3} \cdot \frac{x^3}{1.2.3} + \frac{1}{4} \cdot \frac{x^4}{1.2.3.4} + \text{etc.}$$

Atque hinc etiam ponendo $e^x = z$, vt sit $x = l\,z$, formulam supra memoratam $\frac{\partial z}{l\,z}$ per seriem integrare poterimus, eritque

$$\int \frac{\partial z}{l\,z} = C + l\,l\,z + \frac{l\,z}{1} + \frac{1}{2} \cdot \frac{(l\,z)^2}{1.2} + \frac{1}{3} \cdot \frac{(l\,z)^3}{1.2.3} + \frac{1}{4} \cdot \frac{(l\,z)^4}{1.2.3.4} + \text{etc.}$$

quod integrale si debeat euanescere, sumto $z = 0$, constans C fit infinita, vnde pro reliquis casibus nihil concludi potest. Idem incommodum locum habet, si euanescens reddamus casu $z = 1$, quia $l\,l\,z = l\,0$ fit infinitum. Caeterum patet, si integrale sit reale, pro valoribus ipsius z vnitate minoribus, vbi $l\,z$ est negatiuus, tum pro valoribus vnitate maioribus fieri imaginarium, et viciſſim. Hinc ergo natura huius functionis transcendentis parum cognoscitur.

Scholion 2.

229. Quando vel integratio non succedit, vel series ante inuentae minus idoneae videntur, hinc quantitatem a^x in

seriem resoluendo, statim sine aliis subsidiis formulae $a^x X \partial x$ integrale per seriem exhiberi potest, erit enim

$$\int a^x X \partial x = \int X \partial x + \frac{la}{1}\int X x \partial x + \frac{(la)^2}{1.2}\int X x^2 \partial x$$
$$+ \frac{(la)^3}{1.2.3}\int X x^3 \partial x + \text{etc.}$$

Ita si sit $X = x^n$, habebitur

$$\int a^x x^n \partial x = C + \frac{x^{n+1}}{n+1} + \frac{x^{n+2}\, la}{1(n+2)} + \frac{x^{n+3}(la)^2}{1.2(n+3)}$$
$$+ \frac{x^{n+4}(la)^3}{1.2.3(n+4)} + \text{etc.}$$

vbi notantum, si n fuerit numerus integer negatiuus, puta $n = -i$, loco $\frac{x^{n+i}}{n+i}$ scribi debere lx.

Exemplum 3.

230. *Formulae* $\frac{a^x \partial x}{1-x}$ *integrale per seriem infinitam exprimere.*

Per priorem solutionem obtinemus, ob

$$X = \frac{1}{1-x};\ P = \frac{\partial X}{\partial x} = \frac{1}{(1-x)^2};\ Q = \frac{\partial P}{\partial x} = \frac{1.2}{(1-x)^3};\ R = \frac{\partial Q}{\partial x} = \frac{1.2.3}{(1-x)^4}\ \text{etc.}$$

hincque sequentem seriem:

$$\int \frac{a^x \partial x}{1-x} = a^x\left(\frac{1}{(1-x)(la)} - \frac{1}{(1-x)^2(la)^2} + \frac{1.2}{(1-x)^3(la)^3} - \frac{1.2.3}{(1-x)^4(la)^4} + \text{etc.}\right)$$

Aliae series reperiuntur, si vel a^x, vel fractio $\frac{1}{1-x}$ in seriem euoluatur. Commodissima autem videtur, quae seriem fingendo eruitur: breuitatis gratia pro a sumamus numerum e, vt $le = 1$, ac statuatur $\partial y = \frac{e^x \partial x}{1-x}$ seu

$$\frac{\partial y}{\partial x}(1-x) - 1 - x - \frac{x^2}{1.2} - \frac{x^3}{1.2.3} - \frac{x^4}{1.2.3.4} - \text{etc.} = 0.$$

Iam

Iam pro y fingatur haec feries

$$y = \int \frac{e^x \partial x}{1-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

eritque facta fubftitutione

$$\left.\begin{array}{cccccc} & B + & 2Cx + & 3Dx^2 + & 4Ex^3 + & 5Fx^4 + \text{etc.} \\ & - B & - 2C & - 3D & - 4E & \\ -1 & - 1 & - \frac{1}{2} & - \frac{1}{6} & - \frac{1}{24} & \end{array}\right\} = 0:$$

vnde eliciuntur iftae determinationes:

$$\begin{array}{l|l} B = 1 & E = \frac{1}{4}(1 + 1 + \frac{1}{2} + \frac{1}{6}) \\ C = \frac{1}{2}(1 + 1) & F = \frac{1}{5}(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}) \\ D = \frac{1}{3}(1 + 1 + \frac{1}{2}) & \quad \text{etc.} \end{array}$$

Problema 22.

231. *Formulae differentialis* $\partial y = x^{nx} \partial x$ *integrale inveftigare, ac per feriem infinitam exprimere.*

Solutio.

Commodius hoc praeftari nequit, quam vt formula exponentialis x^{nx} in feriem infinitam conuertatur, quae eft

$$x^{nx} = 1 + nxlx + \frac{n^2x^2(lx)^2}{1.2} + \frac{n^3x^3(lx)^3}{1.2.3} + \frac{n^4x^4(lx)^4}{1.2.3.4} + \text{etc.}$$

qua per ∂x multiplicata, et fingulis terminis integratis, erit:

$$\int \partial x = x;$$

$$\int x \partial x\, lx = x^2\left(\frac{lx}{2} - \frac{1}{2^2}\right);$$

$$\int x^2 \partial x (lx)^2 = x^3\left(\frac{(lx)^2}{3} - \frac{2lx}{3^2} + \frac{2.1}{3^3}\right);$$

$$\int x^3 \partial x (lx)^3 = x^4\left(\frac{(lx)^3}{4} - \frac{3(lx)^2}{4^2} + \frac{3.2lx}{4^3} - \frac{3.2.1}{4^4}\right);$$

$$\int x^4 \partial x (lx)^4 = x^5\left(\frac{(lx)^4}{5} - \frac{4(lx)^3}{5^2} + \frac{4.3(lx)^2}{5^3} - \frac{4.3.2lx}{5^4} + \frac{4.3.2.1}{5^5}\right).$$

etc.

Quare fi hae feries fubftituantur, et fecundum poteftates ipfius lx difponantur, integrale quaefitum exprimetur per has innumerabiles feries infinitas:

$y =$

$$y = \int x^{nx}\,\partial x = +x\left(1 - \frac{nx}{2^2} + \frac{n^2x^2}{3^3} - \frac{n^3x^3}{4^4} + \frac{n^4x^4}{5^5} - \text{etc.}\right)$$
$$+ \frac{nx^2 lx}{1}\left(\frac{1}{2^1} - \frac{nx}{3^2} + \frac{n^2x^2}{4^3} - \frac{n^3x^3}{5^4} + \frac{n^4x^4}{6^5} - \text{etc.}\right)$$
$$+ \frac{n^2x^3(lx)^2}{1.2}\left(\frac{1}{3^1} - \frac{nx}{4^2} + \frac{n^2x^2}{5^3} - \frac{n^3x^3}{6^4} + \frac{n^4x^4}{7^5} - \text{etc.}\right)$$
$$+ \frac{n^3x^4(lx)^3}{1.2.3}\left(\frac{1}{4^1} - \frac{nx}{5^2} + \frac{n^2x^2}{6^3} - \frac{n^3x^3}{7^4} + \frac{n^4x^4}{8^5} - \text{etc.}\right)$$
etc.

quod integrale ita eſt ſumtum, vt euaneſcat, poſito $x = 0$.

Corollarium.

232. Hac ergo lege inſtituta integratione, ſi ponatur $x = 1$, valor integralis $\int x^{nx}\,\partial x$ huic ſeriei aequatur

$$1 - \frac{n}{2^2} + \frac{n^2}{3^3} - \frac{n^3}{4^4} + \frac{n^4}{5^5} - \frac{n^5}{6^6} + \text{etc.}$$

quae ob concinnitatem terminorum omnino eſt notatu digna.

Scholion.

233. Eodem modo reperitur integrale huius formulae:

$$y = \int x^{nx} x^m\,\partial x = \int x^m\,\partial x\left(1 + nx\,lx + \frac{n^2x^2(lx)^2}{1.2} + \frac{n^3x^3(lx)^3}{1.2.3} + \text{etc.}\right)$$

erit enim ſingulis terminis integrandis:

$$\int x^m\,\partial x = \frac{x^{m+1}}{m+1};$$
$$\int x^{m+1}\,\partial x\,(lx) = x^{m+2}\left(\frac{lx}{m+2} - \frac{1}{(m+2)^2}\right);$$
$$\int x^{m+2}\,\partial x\,(lx)^2 = x^{m+3}\left(\frac{(lx)^2}{m+3} - \frac{2.lx}{(m+3)^2} + \frac{2.1}{(m+3)^3}\right);$$
$$\int x^{m+3}\,\partial x\,(lx)^3 = x^{m+4}\left(\frac{(lx)^3}{m+4} - \frac{3(lx)^2}{(m+4)^2} + \frac{3.2\,lx}{(m+4)^3} - \frac{3.2.1}{(m+4)^4}\right);$$
etc.

Quod ſi ergo integrale ita determinetur, vt euaneſcat poſito $x = 0$, tum vero ſtatuatur $x = 1$, pro hoc caſu valor formulae integralis $\int x^{nx} x^m\,\partial x$ exprimetur hac ſerie ſatis memorabili:

$$\frac{1}{m+1} - \frac{n}{(m+2)^2} + \frac{nn}{(m+3)^3} - \frac{n^3}{(m+4)^4} + \frac{n^4}{(m+5)^5} - \text{etc.}$$

quae

quae vti manifeſtum eſt, locum habere nequit, quoties m eſt numerus integer negatiuus.

Alia exempla formularum exponentialium non adiungo, quia plerumque integralia nimis inconcinne exprimuntur, methodus autem eas tractandi hic ſufficienter eſt expoſita. Interim tamen ſingularem attentionem merentur formulae integrationem abſolute admittentes, quae in hac forma continentur $e^x(\partial P + P\partial x)$ cuius integrale manifeſto eſt $e^x P$. Huiusmodi autem caſibus difficile eſt regulas tradere integrale inueniendi, et coniecturae plerumque plurimum eſt tribuendum. Veluti ſi proponeretur haec formula $\frac{e^x x\partial x}{(1+x)^2}$, facile eſt ſuſpicari integrale, ſi datur, talem formam eſſe habiturum $\frac{e^x z}{1+x}$. Huius ergo differentiale $\frac{e^x[\partial z(1+x)+xz\partial x]}{(1+x)^2}$ cum illo comparatum dat $\partial z(1+x)+xz\partial x = x\partial x$, vbi ſtatim patet eſſe $z=1$, quod niſi per ſe pateret, ex regulis difficulter cognoſceretur. Quare tranſeo ad alterum genus formularum tranſcendentium iam in Analyſin receptarum, quae vel angulos vel ſinus, tangentesue angulorum complectuntur.

CAPVT V.

DE

INTEGRATIONE FORMVLARVM ANGVLOS SINVSVE ANGVLORVM IMPLICANTIVM.

Problema 23.

234.

Proposita formula differentiali $X \partial x$ Ang. sin. x, eius integrale inuestigare.

Solutio.

Cum sit ∂. Ang. sin. $x = \frac{\partial x}{\sqrt{(1 - xx)}}$, formula proposita, ita in factores discerpatur, Ang. sin. $x \times X \partial x$. Si iam $X \partial x$ integrationem patiatur, sitque $\int X \partial x = P$, erit nostrum integrale $\int X \partial x$ Ang. sin. $x = P$ Ang. sin. $x - \int \frac{P \partial x}{\sqrt{(1 - xx)}}$; itaque opus reductum est ad integrationem formulae algebraicae, pro qua supra praecepta sunt tradita.

Caeterum si fuerit $X = \frac{1}{\sqrt{(1 - xx)}}$, manifestum est integrale fore $\int \frac{\partial x}{\sqrt{(1 - xx)}}$ Ang. sin. $x = \frac{1}{2}$ (Ang. sin. $x)^2$; quo solo casu quadratum anguli in integrale ingreditur.

Exemplum 1.

235. *Hanc formulam* $\partial y = x^n \partial x$ Ang. sin. x *integrare.*

Cum sit $P = \int x^n \partial x = \frac{x^{n+1}}{n+1}$ habebimus

$$y = \frac{x^{n+1}}{n+1} \text{ Ang. sin. } x - \frac{1}{n+1} \int \frac{x^{n+1} \partial x}{\sqrt{(1 - xx)}}.$$

Hinc

Hinc pro variis valoribus ipsius n erunt integralia ope §. 120. eruta, vt sequentur:

$$\int \partial x \text{ Ang. fin. } x = x \text{ Ang. fin. } x + \sqrt{(1 - xx)} - 1;$$

$$\int x \partial x \text{ Ang. fin. } x = \tfrac{1}{2} x x \text{ Ang. fin. } x + \tfrac{1}{4} x \sqrt{(1 - xx)} - \tfrac{1}{4} \text{ Ang. fin. } x;$$

$$\int x^2 \partial x \text{ Ang. fin. } x = \tfrac{1}{3} x^3 \text{ Ang. fin. } x + \tfrac{1}{3}\left(\tfrac{1}{3} x^2 + \tfrac{2}{3}\right) \sqrt{(1 - xx)} - \tfrac{1}{3} \cdot \tfrac{2}{1};$$

$$\int x^3 \partial x \text{ Ang. fin. } x = \tfrac{1}{4} x^4 \text{ Ang. fin. } x + \tfrac{1}{4}\left(\tfrac{1}{4} x^3 + \tfrac{1.3}{2.4} x\right) \sqrt{(1 - xx)} - \tfrac{1}{4} \cdot \tfrac{1.3}{2.4} \text{ Ang. fin. } x;$$

quae ita funt fumta, vt euanefcant pofito $x = 0$.

Exemplum 2.

236. *Hanc formulam* $\partial y = \frac{x \partial x}{\sqrt{(1 - xx)}}$ Ang. fin. x *integrare.*

Cum fit $\int \frac{x \partial x}{\sqrt{(1 - xx)}} = -\sqrt{(1 - xx)} = P$, erit integrale quaefitum $y = C - \sqrt{(1 - xx)}$ Ang. fin. $x + \int \frac{\partial x \sqrt{(1 - xx)}}{\sqrt{(1 - xx)}}$, ficque habebitur:

$$y = \int \frac{x \partial x}{\sqrt{(1 - xx)}} \text{ Ang. fin. } x = C - \sqrt{(1 - xx)} \text{ Ang. fin. } x + x.$$

Exemplum 3.

237. *Hanc formulam* $\partial y = \frac{\partial x}{(1 - xx)^{\frac{3}{2}}}$ Ang. fin. x *integrare.*

Hic eft $P = \int \frac{\partial x}{(1 - xx)^{\frac{3}{2}}} = \frac{x}{\sqrt{(1 - xx)}}$, vnde fit

$y = \frac{x}{\sqrt{(1 - xx)}}$ Ang. fin. $x - \int \frac{x \partial x}{1 - xx}$, feu

$y =$

$$y = \int \frac{\partial x}{(1-xx)^{\frac{3}{2}}} \text{ Ang. fin. } x = \frac{x}{\sqrt{(1-xx)}} \text{ Ang. fin. } x + l\sqrt{(1-xx)},$$

quod integrale euanefcit pofito $x = 0$.

Scholion.

238. Simili modo integratur formula $\partial y = X\partial x$ Ang. cof. x. Cum enim fit ∂. Ang. cof. $x = \frac{-\partial x}{\sqrt{(1-xx)}}$, fi ponamus $\int X\partial x = P$, erit $y = P$ Ang. cof. $x + \int \frac{P\partial x}{\sqrt{(1-xx)}}$. Quin etiam fi proponatur formula $\partial y = X\partial x$ Ang. tang. x, quia eft ∂. Ang. tang. $x = \frac{\partial x}{1+xx}$, pofito $\int X\partial x = P$, erit hoc integrale:

$$y = \int X\partial x \text{ Ang. tang. } x = P \text{ Ang. tang. } x - \int \frac{P\partial x}{1+xx}.$$

Quoties ergo $\int X\partial x$ algebraice dari poteft, toties integratio reducitur ad formulam algebraicam, ficque negotium confectum eft habendum. Cum igitur in his formulis angulus, cuius finus, cofinus, vel tangens erat $= x$, ineffet, confideremus etiam eiusmodi formulas, in quas quadratum huius anguli, altiorue poteftas ingreditur.

Problema 24.

239. Denotet Φ angulum, cuius finus tangensue eft functio quaedam ipfius x, vnde fiat $\partial\Phi = u\partial x$, propofitaque fit haec formula $\partial y = X\partial x . \Phi^n$ quam integrare oporteat.

Solutio.

Sit $\int X\partial x = P$, vt habeamus $\partial y = \Phi^n \partial P$, eritque integrando $y = \Phi^n P - n\int \Phi^{n-1} P u \partial x$. Iam fimili modo fit $\int P u \partial x = Q$, erit

$$\int \Phi^{n-1} P u \partial x = \Phi^{n-1} Q - (n-1)\int \Phi^{n-2} Q u \partial x,$$

tum pofito $\int Q u \partial x = R$, erit

$$\int \Phi^{n-2} Q u \partial x = \Phi^{n-2} R - (n-2)\int \Phi^{n-3} R u \partial x.$$

Hoc-

Hocque modo poteſtas anguli Φ continuo deprimitur, donec tandem ad formulam ab angulo Φ liberam perueniatur: id quod ſemper eueniet, dummodo n ſit numerus integer poſitiuus, et haec integralia continuo ſumere liceat $\int X \partial x = P$, $\int P u \partial x = Q$, $\int Q u \partial x = R$, etc. quae integrationes, ſi non ſuccedant, fruſtra integratio ſuſcipitur.

Exemplum.

240. *Sit Φ angulus cuius ſinus $= x$, vt ſit $\partial \Phi = \frac{\partial x}{\sqrt{(1-xx)}}$, integrare formulam $\partial y = \Phi^n \partial x$.*

Erit ergo $X = 1$,

$$P = x,\ Q = \int \frac{P \partial x}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)},$$

$$R = \int \frac{Q \partial x}{\sqrt{(1-xx)}} = -x,\ S = \int \frac{R \partial x}{\sqrt{(1-xx)}} = \sqrt{(1-xx)},\ T = x, \text{ etc.}$$

quibus valoribus inuentis reperietur:

$$y = \int \Phi^n \partial x = \Phi^n x + n \Phi^{n-1} \sqrt{(1-xx)} - n(n-1) \Phi^{n-2} x$$
$$- n(n-1)(n-2) \Phi^{n-3} \sqrt{(1-xx)} + \text{etc.}$$

Pro variis ergo valoribus exponentis n habebimus:

$$\int \Phi \partial x = \Phi x + \sqrt{(1-xx)} - 1;$$
$$\int \Phi^2 \partial x = \Phi^2 x + 2 \Phi \sqrt{(1-xx)} - 2.1\, x;$$
$$\int \Phi^3 \partial x = \Phi^3 x + 3 \Phi^2 \sqrt{(1-xx)} - 3.2 \Phi x - 3.2.1 \sqrt{(1-xx)} + 6;$$
$$\text{etc.}$$

integralibus ita determinatis, vt euaneſcant poſito $x = 0$.

Scholion.

241. Si ſit $X \partial x = u \partial x = \partial \Phi$, formulae $\Phi^n \partial \Phi$ integrale eſt $\frac{1}{n+1} \Phi^{n+1}$; ſimilique modo, ſi fuerit Φ functio quaecunque anguli Φ, formulae $\Phi u \partial x = \Phi \partial \Phi$ integratio nihil habet difficultatis. Multo latius patent formulae ſinus, coſinusve angulorum et tangentes implicantes, quarum integratio per

 inuer-

inuerſam Analyſin ampliſſimum habet vſum; cum praecipue Theoria Aſtronomiae ad huiusmodi formulas ſit reducta. Prima autem fundamenta peti debent ex calculo differentiali, vnde cum ſit:

$$\partial.\,\text{ſin.}\,n\Phi = n\,\partial\Phi\,\text{coſ.}\,n\Phi;\quad \partial.\,\text{coſ.}\,n\Phi = -n\,\partial\Phi\,\text{ſin.}\,n\Phi;$$

$$\partial.\,\text{tang.}\,n\Phi = \frac{n\,\partial\Phi}{\text{coſ.}\,n\Phi^2};\quad \partial.\,\text{cot.}\,n\Phi = \frac{-n\,\partial\Phi}{\text{ſin.}\,n\Phi^2};$$

$$\partial.\,\frac{1}{\text{ſin.}\,n\Phi} = \frac{-n\,\partial\Phi\,\text{coſ.}\,n\Phi}{\text{ſin.}\,n\Phi^2};\quad \partial.\,\frac{1}{\text{coſ.}\,n\Phi} = \frac{n\,\partial\Phi\,\text{ſin.}\,n\Phi}{\text{coſ.}\,n\Phi^2};$$

nanciſcimur has integrationes elementares:

$$\int\partial\Phi\,\text{coſ.}\,n\Phi = \tfrac{1}{n}\,\text{ſin.}\,n\Phi;\quad \int\partial\Phi\,\text{ſin.}\,n\Phi = -\tfrac{1}{n}\,\text{coſ.}\,n\Phi;$$

$$\int\frac{\partial\Phi}{\text{coſ.}\,n\Phi^2} = \tfrac{1}{n}\,\text{tang.}\,n\Phi;\quad \int\frac{\partial\Phi}{\text{ſin.}\,n\Phi^2} = -\tfrac{1}{n}\,\text{cot.}\,n\Phi;$$

$$\int\frac{\partial\Phi\,\text{coſ.}\,n\Phi}{\text{ſin.}\,n\Phi^2} = -\frac{1}{n\,\text{ſin.}\,n\Phi};\quad \int\frac{\partial\Phi\,\text{ſin.}\,n\Phi}{\text{coſ.}\,n\Phi^2} = \frac{1}{n\,\text{coſ.}\,n\Phi};$$

vnde ſtatim huiusmodi formularum differentialium integratio

$$\partial\Phi(A + B\,\text{coſ.}\,\Phi + C\,\text{coſ.}\,2\Phi + D\,\text{coſ.}\,3\Phi + E\,\text{coſ.}\,4\Phi \text{ etc.})$$

conſequitur, cum integrale manifeſto ſit

$$A\Phi + B\,\text{ſin.}\,\Phi + \tfrac{1}{2}C\,\text{ſin.}\,2\Phi + \tfrac{1}{3}D\,\text{ſin.}\,3\Phi + \tfrac{1}{4}E\,\text{ſin.}\,4\Phi \text{ etc.}$$

Deinde etiam in ſubſidium vocari conuenit, quae in elementis de angulorum compoſitione traduntur: ſcilicet

$$\text{ſin.}\,\alpha.\,\text{ſin.}\,\beta = \tfrac{1}{2}\text{coſ.}(\alpha-\beta) - \tfrac{1}{2}\text{coſ.}(\alpha+\beta);$$

$$\text{coſ.}\,\alpha.\,\text{coſ.}\,\beta = \tfrac{1}{2}\text{coſ.}(\alpha-\beta) + \tfrac{1}{2}\text{coſ.}(\alpha+\beta);$$

$$\text{ſin.}\,\alpha\,\text{coſ.}\,\beta = \tfrac{1}{2}\text{ſin.}(\alpha+\beta) + \tfrac{1}{2}\text{ſin.}(\alpha-\beta) = \tfrac{1}{2}\text{ſin.}(\alpha+\beta) - \tfrac{1}{2}\text{ſin.}(\beta-\alpha);$$

vnde producta plurium ſinuum et coſinuum in ſimplices ſinus coſinusue reſoluuntur.

Problema 5.

242. **Formulae $\partial\Phi\,\text{ſin.}\,\Phi^n$ integrale inueſtigare.**

Solu-

Solutio.

Repraesentetur in hos factores resoluta $\sin.\Phi^{n-1}.\partial\Phi\sin.\Phi$, et quia $\int\partial\Phi\sin.\Phi = -\cos.\Phi$, erit

$$\int\partial\Phi\sin.\Phi^n = -\sin.\Phi^{n-1}\cos.\Phi + (n-1)\int\partial\Phi\sin.\Phi^{n-2}\cos.\Phi^2.$$

Hinc ob $\cos.\Phi^2 = 1 - \sin.\Phi^2$, habebitur

$$\int\partial\Phi\sin.\Phi^n = -\sin.\Phi^{n-1}\cos.\Phi$$
$$+(n-1)\int\partial\Phi\sin.\Phi^{n-2} - (n-1)\int\partial\Phi\sin.\Phi^n:$$

vbi cum postrema formula ipsi propositae sit similis, hinc colligitur ista reductio

$$\int\partial\Phi\sin.\Phi^n = -\tfrac{1}{n}\sin.\Phi^{n-1}\cos.\Phi + \tfrac{n-1}{n}\int\partial\Phi\sin.\Phi^{n-2},$$

qua integratio ad hanc formulam simpliciorem $\partial\Phi\sin.\Phi^{n-2}$ reuocatur. Cum igitur casus simplicissimi constent,

$$\int\partial\Phi\sin.\Phi^0 = \Phi \text{ et } \int\partial\Phi\sin.\Phi = -\cos.\Phi,$$

hinc via ad continuo maiores exponentes n paratur:

$$\int\partial\Phi\sin.\Phi^0 = \Phi$$
$$\int\partial\Phi\sin.\Phi = -\cos.\Phi$$
$$\int\partial\Phi\sin.\Phi^2 = -\tfrac{1}{2}\sin.\Phi\cos.\Phi + \tfrac{1}{2}\Phi$$
$$\int\partial\Phi\sin.\Phi^3 = -\tfrac{1}{3}\sin.\Phi^2\cos.\Phi - \tfrac{2}{3}\cos.\Phi$$
$$\int\partial\Phi\sin.\Phi^4 = -\tfrac{1}{4}\sin.\Phi^3\cos.\Phi - \tfrac{1.3}{2.4}\sin.\Phi\cos.\Phi + \tfrac{1.3}{2.4}\Phi$$
$$\int\partial\Phi\sin.\Phi^5 = -\tfrac{1}{5}\sin.\Phi^4\cos.\Phi - \tfrac{1.4}{3.5}\sin.\Phi^2\cos.\Phi - \tfrac{2.4}{3.5}\cos.\Phi$$
$$\int\partial\Phi\sin.\Phi^6 = -\tfrac{1}{6}\sin.\Phi^5\cos.\Phi - \tfrac{1.5}{4.6}\sin.\Phi^3\cos.\Phi$$
$$-\tfrac{1.3.5}{2.4.6}\sin.\Phi\cos.\Phi + \tfrac{1.3.5}{2.4.6}\Phi$$

etc.

Corollarium 1.

243. Quoties n est numerus impar, integrale per solum sinum et cosinum exhibetur, at si n est numerus par, integrale

tegrale insuper ipsum angulum inuoluit, ideoque est functio transcendens.

Corollarium 2.

244. Casibus ergo quibus n est numerus impar, id imprimis notari conuenit; etiamsi angulus seu arcus Φ in infinitum crescat, integrale tamen nunquam vltra certum limitem excrescere posse, cum tamen si n sit numerus par, etiam in infinitum excrescat.

Scholion.

245. Simili modo formula $\partial\Phi \cos.\Phi^n$ tractatur, quae in hos factores resoluta $\cos.\Phi^{n-1}.\partial\Phi\cos.\Phi$, praebet

$$\int\partial\Phi\cos.\Phi^n = \cos.\Phi^{n-1}\sin.\Phi + (n-1)\int\partial\Phi\cos.\Phi^{n-2}\sin.\Phi^2$$
$$= \cos.\Phi^{n-1}\sin.\Phi + (n-1)\int\partial\Phi\cos.\Phi^{n-2} - (n-1)\int\partial\Phi\cos.\Phi^n:$$

vnde cum postrema formula propositae sit similis, colligitur

$$\int\partial\Phi\cos.\Phi^n = \tfrac{1}{n}\sin.\Phi\cos.\Phi^{n-1} + \tfrac{n-1}{n}\int\partial\Phi\cos.\Phi^{n-2}.$$

Quare cum casibus $n=0$, et $n=1$ integratio sit in promptu, ad altiores potestates patet progressio:

$$\int\partial\Phi\cos.\Phi^0 = \Phi$$
$$\int\partial\Phi\cos.\Phi = \sin.\Phi$$
$$\int\partial\Phi\cos.\Phi^2 = \tfrac{1}{2}\sin.\Phi\cos.\Phi + \tfrac{1}{2}\Phi$$
$$\int\partial\Phi\cos.\Phi^3 = \tfrac{1}{3}\sin.\Phi\cos.\Phi^2 + \tfrac{2}{3}\sin.\Phi$$
$$\int\partial\Phi\cos.\Phi^4 = \tfrac{1}{4}\sin.\Phi\cos.\Phi^3 + \tfrac{1.3}{2.4}\sin.\Phi\cos.\Phi + \tfrac{1.3}{2.4}\Phi$$
$$\int\partial\Phi\cos.\Phi^5 = \tfrac{1}{5}\sin.\Phi\cos.\Phi^4 + \tfrac{1.4}{3.5}\sin.\Phi\cos.\Phi^2 + \tfrac{2.4}{3.5}\sin.\Phi$$
$$\int\partial\Phi\cos.\Phi^6 = \tfrac{1}{6}\sin.\Phi\cos.\Phi^5 + \tfrac{1.5}{4.6}\sin.\Phi\cos.\Phi^3 + \tfrac{1.3.5}{2.4.6}\sin.\Phi\cos.\Phi + \tfrac{1.3.5}{2.4.6}\Phi$$

etc.

Pro-

Problema 26.

246. Formulae $\partial\Phi \sin.\Phi^m \cos.\Phi^n$ integrale inuenire.

Solutio.

Quo hoc facilius praestetur, consideremus factum $\sin.\Phi^\mu \cos.\Phi^\nu$, quod differentiatum fit $\mu\,\partial\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu+1} - \nu\,\partial\Phi \sin.\Phi^{\mu+1}\cos.\Phi^{\nu-1}$. Iam prout vel in parte priori $\cos.\Phi^2 = 1 - \sin.\Phi^2$, vel in posteriori $\sin.\Phi^2 = 1 - \cos.\Phi^2$ statuitur, oritur

$$\text{vel } \partial.\sin.\Phi^\mu \cos.\Phi^\nu = \mu\,\partial\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1} - (\mu+\nu)\,\partial\Phi \sin.\Phi^{\mu+1}\cos.\Phi^{\nu-1},$$

$$\text{vel } \partial.\sin.\Phi^\mu \cos.\Phi^\nu = -\nu\,\partial\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1} + (\mu+\nu)\,\partial\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu+1}.$$

Hinc igitur duplicem reductionem adipiscimur:

$$\text{I. } \int\partial\Phi \sin.\Phi^{\mu+1}\cos.\Phi^{\nu-1} = -\frac{1}{\mu+\nu}\sin.\Phi^\mu \cos.\Phi^\nu + \frac{\mu}{\mu+\nu}\int\partial\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1}$$

$$\text{II. } \int\partial\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu+1} = \frac{1}{\mu+\nu}\sin.\Phi^\mu \cos.\Phi^\nu + \frac{\nu}{\mu+\nu}\int\partial\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1}$$

Quare formula proposita $\int\partial\Phi \sin.\Phi^m \cos.\Phi^n$ successiue continuo ad simpliciores potestates tam ipsius $\sin.\Phi$ quam ipsius $\cos.\Phi$ reducitur, donec alter vel penitus abeat, vel simpliciter adsit, quo casu integratio per se patet, cum sit

$$\int\partial\Phi \sin.\Phi^m \cos.\Phi = +\frac{1}{m+1}\sin.\Phi^{m+1} \text{ et}$$
$$\int\partial\Phi \sin.\Phi \cos.\Phi^n = -\frac{1}{n+1}\cos.\Phi^{n+1}.$$

Exemplum.

247. *Formulae* $\partial\Phi \sin.\Phi^8 \cos.\Phi^7$ *integrale inuenire.*

 Per

Per priorem reductionem ob $\mu = 7$ et $\nu = 8$, impetramus

$$\int \partial \Phi \,\text{sin.}\, \Phi^8 \,\text{cos.}\, \Phi^7 = -\tfrac{1}{15} \,\text{sin.}\, \Phi^7 \,\text{cos.}\, \Phi^8 + \tfrac{7}{15} \int \partial \Phi \,\text{sin.}\, \Phi^6 \,\text{cos.}\, \Phi^7;$$

istam per posteriorem reductionem tractemus:

$$\int \partial \Phi \,\text{sin.}\, \Phi^6 \,\text{cos.}\, \Phi^7 = \tfrac{1}{13} \,\text{sin.}\, \Phi^7 \,\text{cos.}\, \Phi^6 + \tfrac{6}{13} \int \partial \Phi \,\text{sin.}\, \Phi^6 \,\text{cos.}\, \Phi^5,$$

hoc modo vlterius progrediamur:

$$\int \partial \Phi \,\text{sin.}\, \Phi^6 \,\text{cos.}\, \Phi^5 = -\tfrac{1}{11} \,\text{sin.}\, \Phi^5 \,\text{cos.}\, \Phi^6 + \tfrac{5}{11} \int \partial \Phi \,\text{sin.}\, \Phi^4 \,\text{cos.}\, \Phi^5$$

$$\int \partial \Phi \,\text{sin.}\, \Phi^4 \,\text{cos.}\, \Phi^5 = \tfrac{1}{9} \,\text{sin.}\, \Phi^5 \,\text{cos.}\, \Phi^4 + \tfrac{4}{9} \int \partial \Phi \,\text{sin.}\, \Phi^4 \,\text{cos.}\, \Phi^3$$

$$\int \partial \Phi \,\text{sin.}\, \Phi^4 \,\text{cos.}\, \Phi^3 = -\tfrac{1}{7} \,\text{sin.}\, \Phi^3 \,\text{cos.}\, \Phi^4 + \tfrac{3}{7} \int \partial \Phi \,\text{sin.}\, \Phi^2 \,\text{cos.}\, \Phi^3$$

$$\int \partial \Phi \,\text{sin.}\, \Phi^2 \,\text{cos.}\, \Phi^3 = \tfrac{1}{5} \,\text{sin.}\, \Phi^3 \,\text{cos.}\, \Phi^2 + \tfrac{2}{5} \int \partial \Phi \,\text{sin.}\, \Phi^2 \,\text{cos.}\, \Phi$$

$$\int \partial \Phi \,\text{sin.}\, \Phi^2 \,\text{cos.}\, \Phi = -\tfrac{1}{3} \,\text{sin.}\, \Phi \,\text{cos.}\, \Phi^2 + \tfrac{1}{3} \int \partial \Phi \,\text{cos.}\, \Phi \left(+\tfrac{1}{3} \,\text{sin.}\, \Phi\right).$$

Ex his colligitur formulae propositae integrale

$$\int \partial \Phi \,\text{sin.}\, \Phi^8 \,\text{cos.}\, \Phi^7$$

$$\begin{aligned} = &-\tfrac{1}{15} \,\text{sin.}\, \Phi^7 \,\text{cos.}\, \Phi^8 + \frac{1.7}{15.13} \,\text{sin.}\, \Phi^7 \,\text{cos.}\, \Phi^6 - \frac{1.7.6}{15.13.11} \,\text{sin.}\, \Phi^5 \,\text{cos.}\, \Phi^6 \\ &+ \frac{1.7.6.5}{15.13.11.9} \,\text{sin.}\, \Phi^5 \,\text{cos.}\, \Phi^4 - \frac{1.7.5.6.4}{15.13.11.9.7} \,\text{sin.}\, \Phi^3 \,\text{cos.}\, \Phi^4 \\ &+ \frac{1.7.6.5.4.3}{15.13.11.9.7.5} \,\text{sin.}\, \Phi^3 \,\text{cos.}\, \Phi^2 - \frac{1.7.6.5.4.3.2}{15.13.11.9.7.5.3} \,\text{sin.}\, \Phi \,\text{cos.}\, \Phi^2 \\ &+ \frac{1.7.6.5.4.3.2}{15.13.11.9.7.5.3} \,\text{sin.}\, \Phi. \end{aligned}$$

Scholion.

248. Quando autem huiusmodi casus occurunt, semper praestat productum $\text{sin.}\, \Phi^m \,\text{cos.}\, \Phi^n$ in sinus vel cosinus angulorum multiplorum resoluere, quo facto singulae partes facillime integrantur. Caeterum hic breuitatis gratia angulum simpliciter littera Φ indicaui, nihiloque res foret generalior, si per $\alpha \Phi + \beta$ exprimeretur, quemadmodum etiam ante haec expressio Ang. sin. x aeque late patet, ac sit loco x, functio quaecunque scriberetur. Contemplemur ergo eiusmodi formulas, in quibus sinus cosinusue denominatorem occupant, vbi quidem simplicissimae sunt.

$$\text{I. } \frac{\partial\Phi}{\sin.\Phi};\ \text{II. } \frac{\partial\Phi}{\cos.\Phi};\ \text{III. } \frac{\partial\Phi \cos.\Phi}{\sin.\Phi};\ \text{IV. } \frac{\partial\Phi \sin.\Phi}{\cos.\Phi};$$

quarum integralia imprimis nosse oportet. Pro prima adhibeantur hae transformationes

$$\frac{\partial\Phi}{\sin.\Phi} = \frac{\partial\Phi \sin.\Phi}{\sin.\Phi^2} = \frac{\partial\Phi \sin.\Phi}{1-\cos.\Phi^2} = \frac{-\partial x}{1-xx} \text{ (posito } \cos.\Phi = x),$$

vnde fit

$$\int \frac{\partial\Phi}{\sin.\Phi} = -\tfrac{1}{2}\, l\, \frac{1+x}{1-x} = -\tfrac{1}{2}\, l\, \frac{1+\cos.\Phi}{1-\cos.\Phi}.$$

Pro secunda

$$\frac{\partial\Phi}{\cos.\Phi} = \frac{\partial\Phi \cos.\Phi}{\cos.\Phi^2} = \frac{\partial\Phi \cos.\Phi}{1-\sin.\Phi^2} = \frac{\partial x}{1-xx} \text{ (posito } \sin.\Phi = x),$$

ergo

$$\int \frac{\partial\Phi}{\cos.\Phi} = \tfrac{1}{2}\, l\, \frac{1+x}{1-x} = \tfrac{1}{2}\, l\, \frac{1+\sin.\Phi}{1-\sin.\Phi}.$$

Tertiae et quartae integratio manifesto logarithmis conficitur: quare haec integralia probe notasse iuuabit

$$\text{I. } \int \frac{\partial\Phi}{\sin.\Phi} = -\tfrac{1}{2}\, l\, \frac{1+\cos.\Phi}{1-\cos.\Phi} = l\, \frac{\sqrt{(1-\cos.\Phi)}}{\sqrt{(1+\cos.\Phi)}} = l \text{ tang. } \tfrac{1}{2}\Phi,$$

$$\text{II. } \int \frac{\partial\Phi}{\cos.\Phi} = \tfrac{1}{2}\, l\, \frac{1+\sin.\Phi}{1-\sin.\Phi} = l\, \frac{\sqrt{(1+\sin.\Phi)}}{\sqrt{(1-\sin.\Phi)}} = l \text{ tang. } (45^\circ + \tfrac{1}{2}\Phi),$$

$$\text{III. } \int \frac{\partial\Phi \cos.\Phi}{\sin.\Phi} = l \sin.\Phi = \int \frac{\partial\Phi}{tang.\Phi} = \int \partial\Phi \cot.\Phi$$

$$\text{IV. } \int \frac{\partial\Phi \sin.\Phi}{\cos.\Phi} = -l \cos.\Phi = \int \partial\Phi \text{ tang. } \Phi$$

hincque sequitur III. + IV.

$$\int \frac{\partial\Phi}{\sin.\Phi \cos.\Phi} = l\, \frac{\sin.\Phi}{\cos.\Phi} = l \text{ tang. } \Phi.$$

Problema 27.

249. Formularum $\dfrac{\partial\Phi \sin.\Phi^m}{\cos.\Phi^n}$ et $\dfrac{\partial\Phi \cos.\Phi^m}{\sin.\Phi^n}$ integralia inuestigare.

Solutio.

Primo statim perspicitur, alteram formulam in alteram transmutari, posito $\Phi = 90^\circ - \psi$, quia tum fit $\sin.\Phi = \cos.\psi$

et cof. $\Phi = +$ fin. ψ, dummodo notetur fore $\partial\Phi = -\partial\psi$. Quare fufficit priorem tantum tractaffe. Reductio autem prior §. 246. data, fumto $\mu + 1 = m$ et $\nu - 1 = -n$, praebet

$$\int\frac{\partial\Phi\,\text{fin.}\,\Phi^m}{\text{cof.}\,\Phi^n} = -\frac{1}{m-n}\cdot\frac{\text{fin.}\,\Phi^{m-1}}{\text{cof.}\,\Phi^{n-1}} + \frac{m-1}{m-n}\int\frac{\partial\Phi\,\text{fin.}\,\Phi^{m-2}}{\text{cof.}\,\Phi^n}$$

quo pacto in numeratore exponens ipfius fin. Φ continuo binario deprimitur, ita vt tandem perueniatur vel ad $\int\frac{\partial\Phi}{\text{cof.}\,\Phi^n}$ vel ad $\int\frac{\partial\Phi\,\text{fin.}\,\Phi}{\text{cof.}\,\Phi^n} = \frac{1}{(n-1)\,\text{cof.}\,\Phi^{n-1}}$: ideoque fola formula $\int\frac{\partial\Phi}{\text{cof.}\,\Phi^n}$ tractanda fuperfit. Altera autem reductio ibidem tradita (246.) fumto $\mu - 1 = m$ et $\nu - 1 = -n$, dat

$$\int\frac{\partial\Phi\,\text{fin.}\,\Phi^m}{\text{cof.}\,\Phi^{n-2}} = \frac{1}{m-n+2}\cdot\frac{\text{fin.}\,\Phi^{m+1}}{\text{cof.}\,\Phi^{n-1}} - \frac{n-1}{m-n+2}\int\frac{\partial\Phi\,\text{fin.}\,\Phi^m}{\text{cof.}\,\Phi^n}:$$

vnde colligitur

$$\int\frac{\partial\Phi\,\text{fin.}\,\Phi^m}{\text{cof.}\,\Phi^n} = \frac{1}{n-1}\cdot\frac{\text{fin.}\,\Phi^{m+1}}{\text{cof.}\,\Phi^{n-1}} - \frac{m-n+2}{n-1}\int\frac{\partial\Phi\,\text{fin.}\,\Phi^m}{\text{cof.}\,\Phi^{n-2}},$$

cuius reductionis ope exponens ipfius cof. Φ in denominatore continuo binario deprimitur, ita vt tandem vel ad $\int\partial\Phi\,\text{fin.}\,\Phi^m$ vel ad $\int\frac{\partial\Phi\,\text{fin.}\,\Phi^m}{\text{cof.}\,\Phi}$ perueniatur. Illius integratio iam fupra eft monftrata, huius vero forma fi $m > 1$, per priorem reductionem tandem vel ad $\int\frac{\partial\Phi}{\text{cof.}\,\Phi}$ vel ad $\int\frac{\partial\Phi\,\text{fin.}\,\Phi}{\text{cof.}\,\Phi}$ reuocatur, illius autem integrale eft $l\,\text{tang.}\,(45° + \frac{1}{2}\Phi)$, huius vero $-l\,\text{cof.}\,\Phi$.

Corollarium 1.

250. Prior reductio non habet locum, quoties eft $m = n$,

$m = n$, hoc scilicet casu formula $\int \frac{\partial \Phi \sin. \Phi^n}{\cos. \Phi^n}$ non reduci potest ad formulam $\int \frac{\partial \Phi \sin. \Phi^{n-2}}{\cos. \Phi^n}$. Altera autem reductione semper vti licet, etsi enim casus $n = 1$ inde excluditur, eius tamen integratio per priorem effici potest.

Corollarium 2.

251. Ratio autem illius exclusionis in hoc est posita, quod formula $\int \frac{\partial \Phi \sin. \Phi^{n-2}}{\cos. \Phi^n}$ est absolute integrabilis, habens integrale $= \frac{1}{n-1} \cdot \frac{\sin. \Phi^{n-1}}{\cos. \Phi^{n-1}}$. Erit ergo pro his casibus:

$$\int \frac{\partial \Phi}{\cos. \Phi^2} = \frac{\sin. \Phi}{\cos. \Phi} = \text{tang.}\, \Phi;$$
$$\int \frac{\partial \Phi \sin. \Phi}{\cos. \Phi^3} = \frac{1}{2} \cdot \frac{\sin. \Phi^2}{\cos. \Phi^2} = \frac{1}{2} \text{tang.}\, \Phi^2;$$
$$\int \frac{\partial \Phi \sin. \Phi^2}{\cos. \Phi^4} = \frac{1}{3} \cdot \frac{\sin. \Phi^3}{\cos. \Phi^3} = \frac{1}{3} \text{tang.}\, \Phi^3;$$
$$\int \frac{\partial \Phi \sin. \Phi^3}{\cos. \Phi^5} = \frac{1}{4} \cdot \frac{\sin. \Phi^4}{\cos. \Phi^4} = \frac{1}{4} \text{tang.}\, \Phi^4.$$

Exemplum 1.

252. *Formulae* $\frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi}$ *integrale assignare.*

Prior reductio dat:

$$\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi} = \frac{-1}{m-1} \sin. \Phi^{m-1} + \int \frac{\partial \Phi \sin. \Phi^{m-2}}{\cos. \Phi}.$$

Hinc a casibus per se notis incipiendo, habebimus:

$$\int \frac{\partial \Phi}{\cos. \Phi} = l\, \text{tang.}\, (45^\circ + \tfrac{1}{2} \Phi)$$
$$\int \frac{\partial \Phi \sin. \Phi}{\cos. \Phi} = -l \cos. \Phi = l \sec. \Phi$$

$$\int \frac{\partial \Phi \sin. \Phi^2}{\cos. \Phi} = -\sin. \Phi + \int \frac{\partial \Phi}{\cos. \Phi}$$
$$\int \frac{\partial \Phi \sin. \Phi^3}{\cos. \Phi} = -\tfrac{1}{2} \sin. \Phi^2 + l \sec. \Phi$$
$$\int \frac{\partial \Phi \sin. \Phi^4}{\cos. \Phi} = -\tfrac{1}{3} \sin. \Phi^3 - \sin. \Phi + \int \frac{\partial \Phi}{\cos. \Phi}$$
$$\int \frac{\partial \Phi \sin. \Phi^5}{\cos. \Phi} = -\tfrac{1}{4} \sin. \Phi^4 - \tfrac{1}{2} \sin. \Phi^2 + l \sec. \Phi$$
$$\int \frac{\partial \Phi \sin. \Phi^6}{\cos. \Phi} = -\tfrac{1}{5} \sin. \Phi^5 - \tfrac{1}{3} \sin. \Phi^3 - \sin. \Phi + \int \frac{\partial \Phi}{\cos. \Phi}$$
$$\int \frac{\partial \Phi \sin. \Phi^7}{\cos. \Phi} = -\tfrac{1}{6} \sin. \Phi^6 - \tfrac{1}{4} \sin. \Phi^4 - \tfrac{1}{2} \sin. \Phi^2 + l \sec. \Phi$$

etc.

Scholion.

253. Pro reliquis casibus denominatoris totum negotium conficietur his reductionibus:

$$\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^2} = \frac{\sin. \Phi^{m+1}}{\cos. \Phi} - m \int \partial \Phi \sin. \Phi^m$$
$$\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^3} = \tfrac{1}{2} \cdot \frac{\sin. \Phi^{m+1}}{\cos. \Phi^2} - \frac{m-1}{2} \int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi}$$
$$\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^4} = \tfrac{1}{3} \cdot \frac{\sin. \Phi^{m+1}}{\cos. \Phi^3} - \frac{m-2}{3} \int \frac{\partial \Phi \cos. \Phi^m}{\cos. \Phi^2}$$
$$\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^5} = \tfrac{1}{4} \cdot \frac{\sin. \Phi^{m+1}}{\cos. \Phi^4} - \frac{m-3}{4} \int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^3}$$

etc.

Exemplum 2.

254. *Formulae* $\frac{\partial \Phi}{\cos. \Phi^n}$ *integrale assignare.*

Altera reductio ob $m = 0$ fit

$$\int \frac{\partial \Phi}{\cos. \Phi^n} = \frac{1}{n-1} \cdot \frac{\sin. \Phi}{\cos. \Phi^{n-1}} + \frac{n-2}{n-1} \int \frac{\partial \Phi}{\cos. \Phi^{n-2}}:$$

quia

quia iam casus simplicissimi $\int \partial \Phi = \Phi$ et

$$\int \frac{\partial \Phi}{\cos. \Phi} = l \operatorname{tang.} (45^\circ + \tfrac{1}{2} \Phi)$$

sunt cogniti, ad eos sequentes omnes reuocabuntur:

$$\int \frac{\partial \Phi}{\cos. \Phi^2} = \frac{\sin. \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi}{\cos. \Phi^3} = \frac{1}{2} \cdot \frac{\sin. \Phi}{\cos. \Phi^2} + \frac{1}{2} \int \frac{\partial \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi}{\cos. \Phi^4} = \frac{1}{3} \cdot \frac{\sin. \Phi}{\cos. \Phi^3} + \frac{2}{3} \frac{\sin. \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi}{\cos. \Phi^5} = \frac{1}{4} \cdot \frac{\sin. \Phi}{\cos. \Phi^4} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\sin. \Phi}{\cos. \Phi^2} + \frac{1 \cdot 3}{2 \cdot 4} \int \frac{\partial \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi}{\cos. \Phi^6} = \frac{1}{5} \cdot \frac{\sin. \Phi}{\cos. \Phi^5} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{\sin. \Phi}{\cos. \Phi^3} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{\sin. \Phi}{\cos. \Phi}.$$

Corollarium 1.

255. Simili modo habebimus has integrationes:

$$\int \frac{\partial \Phi}{\sin. \Phi} = l \operatorname{tang.} \tfrac{1}{2} \Phi; \quad \int \frac{\partial \Phi}{\sin. \Phi^2} = - \frac{\cos. \Phi}{\sin. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^3} = - \frac{1}{2} \cdot \frac{\cos. \Phi}{\sin. \Phi^2} + \frac{1}{2} \int \frac{\partial \Phi}{\sin. \Phi}$$

$$\int \frac{\partial \Phi}{\sin. \Phi^4} = - \frac{1}{3} \cdot \frac{\cos. \Phi}{\sin. \Phi^3} - \frac{2}{3} \cdot \frac{\cos. \Phi}{\sin. \Phi}$$

$$\int \frac{\partial \Phi}{\sin. \Phi^5} = - \frac{1}{4} \cdot \frac{\cos. \Phi}{\sin. \Phi^4} - \frac{1 \cdot 3}{2 \cdot 4} \frac{\cos. \Phi}{\sin. \Phi^2} + \frac{1 \cdot 3}{2 \cdot 4} \int \frac{\partial \Phi}{\sin. \Phi} \text{ etc.}$$

Corollarium 2.

256. Deinde est

$$\int \frac{\partial \Phi \sin. \Phi}{\cos. \Phi^n} = \frac{1}{n-1} \cdot \frac{1}{\cos. \Phi^{n-1}}; \text{ et}$$

$$\int \frac{\partial \Phi \cos. \Phi}{\sin. \Phi^n} = \frac{-1}{n-1} \cdot \frac{1}{\sin. \Phi^{n-1}}.$$

Porro

$$\int \frac{\partial \Phi \sin. \Phi^2}{\cos. \Phi^n} = \int \frac{\partial \Phi}{\cos. \Phi^n} - \int \frac{\partial \Phi}{\cos. \Phi^{n-2}};$$

$$\int \frac{\partial \Phi \cos. \Phi^2}{\sin. \Phi^n} = \int \frac{\partial \Phi}{\sin. \Phi^n} - \int \frac{\partial \Phi}{\sin. \Phi^{n-2}} \text{ et}$$

$\int \partial \Phi$

$$\int \frac{\partial \Phi \,\text{fin.}\, \Phi^3}{\text{fin.}\, \Phi^n} = \int \frac{\partial \Phi \,\text{fin.}\, \Phi}{\text{cof.}\, \Phi^n} - \int \frac{\partial \Phi \,\text{fin.}\, \Phi}{\text{cof.}\, \Phi^{n-2}};$$

$$\int \frac{\partial \Phi \,\text{cof.}\, \Phi^3}{\text{fin.}\, \Phi^n} = \int \frac{\partial \Phi \,\text{cof.}\, \Phi}{\text{fin.}\, \Phi^n} - \int \frac{\partial \Phi \,\text{cof.}\, \Phi}{\text{fin.}\, \Phi^{n-2}};$$

quibus reductionibus continuo vlterius progredi licet.

Problema 28.

257. Formulae $\frac{\partial \Phi}{\text{fin.}\, \Phi^m \,\text{cof.}\, \Phi^n}$ integrale inueſtigare.

Solutio.

Reductiones ſupra adhibitas huc accommodare licet, ſumendo in praecedente problemate m negatiue: ita erit

$$\int \frac{\partial \Phi}{\text{fin.}\, \Phi^m \,\text{cof.}\, \Phi^n} = + \frac{1}{m+n} \cdot \frac{1}{\text{fin.}\, \Phi^{m+1} \,\text{cof.}\, \Phi^{n-1}} + \frac{m+1}{m+n} \int \frac{\partial \Phi}{\text{fin.}\, \Phi^{m+2} \,\text{cof.}\, \Phi^n},$$

vnde loco m ſcribendo $m-2$, per conuerſionem fit

$$\int \frac{\partial \Phi}{\text{fin.}\, \Phi^m \,\text{cof.}\, \Phi^n} = - \frac{1}{m-1} \cdot \frac{1}{\text{fin.}\, \Phi^{m-1} \,\text{cof.}\, \Phi^{n-1}} + \frac{m+n-2}{m-1} \int \frac{\partial \Phi}{\text{fin.}\, \Phi^{m-2} \,\text{cof.}\, \Phi^n}.$$

Altera huic ſimilis eſt

$$\int \frac{\partial \Phi}{\text{fin.}\, \Phi^m \,\text{cof.}\, \Phi^n} = \frac{1}{n-1} \cdot \frac{1}{\text{fin.}\, \Phi^{m-1} \,\text{cof.}\, \Phi^{n-1}} + \frac{m+n-2}{n-1} \int \frac{\partial \Phi}{\text{fin.}\, \Phi^m \,\text{cof.}\, \Phi^{n-2}}.$$

Cum iam in hoc genere formae ſimpliciſſimae ſint:

$\int \partial \Phi$

$\int \frac{\partial \Phi}{\sin. \Phi} = l \text{ tang. } \frac{1}{2} \Phi$; $\int \frac{\partial \Phi}{\cos. \Phi} = l \text{ tang. } (45^\circ + \frac{1}{2} \Phi)$;

$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi} = l \text{ tang.} \Phi$; $\int \frac{\partial \Phi}{\sin. \Phi^2} = - \text{cot.} \Phi$; $\int \frac{\partial \Phi}{\cos. \Phi^2} = \text{tang.} \Phi$;

hinc magis compositas eliciemus:

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^2} = \frac{1}{\cos. \Phi} + \int \frac{\partial \Phi}{\sin. \Phi}; \quad \int \frac{\partial \Phi}{\sin. \Phi^2 \cos. \Phi} = - \frac{1}{\sin. \Phi} + \int \frac{\partial \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^4} = \frac{1}{3} \cdot \frac{1}{\cos. \Phi^3} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^2};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^4 \cos. \Phi} = - \frac{1}{3} \cdot \frac{1}{\sin. \Phi^3} + \int \frac{\partial \Phi}{\sin. \Phi^2 \cos. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^6} = \frac{1}{5} \cdot \frac{1}{\cos. \Phi^5} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^4};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^6 \cos. \Phi} = - \frac{1}{5} \cdot \frac{1}{\sin. \Phi^5} + \int \frac{\partial \Phi}{\sin. \Phi^4 \cos. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^3} = \frac{1}{2} \cdot \frac{1}{\cos. \Phi^2} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^3 \cos. \Phi} = - \frac{1}{2} \cdot \frac{1}{\sin. \Phi^2} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^5} = \frac{1}{4} \cdot \frac{1}{\cos. \Phi^4} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^1};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^5 \cos. \Phi} = - \frac{1}{4} \cdot \frac{1}{\sin. \Phi^4} + \int \frac{\partial \Phi}{\sin. \Phi^3 \cos. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^7} = \frac{1}{6} \cdot \frac{1}{\cos. \Phi^6} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^5};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^7 \cos. \Phi} = - \frac{1}{6} \cdot \frac{1}{\sin. \Phi^6} + \int \frac{\partial \Phi}{\sin. \Phi^5 \cos. \Phi};$$

etc.

$$\int \frac{\partial \Phi}{\sin. \Phi^2 \cos. \Phi^2} = \frac{1}{\sin. \Phi \cos. \Phi} + 2 \int \frac{\partial \Phi}{\sin. \Phi^2} = - \frac{1}{\sin. \Phi \cos. \Phi} + 2 \int \frac{\partial \Phi}{\cos. \Phi^2}$$

$$\int \frac{\partial \Phi}{\sin. \Phi^2 \cos. \Phi^4} = \frac{1}{3} \frac{1}{\sin. \Phi \cos. \Phi^3} + \frac{4}{3} \int \frac{\partial \Phi}{\sin. \Phi^2 \cos. \Phi^2}$$

$$\int \frac{\partial \Phi}{\sin. \Phi^4 \cos. \Phi^2} = - \frac{1}{3} \cdot \frac{1}{\sin. \Phi^3 \cos. \Phi} + \frac{4}{3} \int \frac{\partial \Phi}{\sin. \Phi^2 \cos. \Phi^2}.$$

Sicque formulae quantumuis compositae ad simpliciores, quarum integratio est in promtu, reducuntur.

Corollarium 1.

258. Ambo exponentes ipsius sin. Φ et cos. Φ simul binario minui possunt: erit enim per priorem reductionem

$$\int \frac{\partial \Phi}{\sin. \Phi^{\mu} \cos. \Phi^{\nu}} = -\frac{1}{\mu - 1} \cdot \frac{1}{\sin. \Phi^{\mu - 1} \cos. \Phi^{\nu - 1}}$$
$$+ \frac{\mu + \nu - 2}{\mu - 1} \int \frac{\partial \Phi}{\sin. \Phi^{\mu - 2} \cos. \Phi^{\nu}} :$$

nunc haec formula per posteriorem ob $m = \mu - 2$ et $n = \nu$ dat

$$\int \frac{\partial \Phi}{\sin. \Phi^{\mu - 2} \cos. \Phi^{\nu}} = \frac{1}{\nu - 1} \cdot \frac{1}{\sin. \Phi^{\mu - 3} \cos. \Phi^{\nu - 1}}$$
$$+ \frac{\mu + \nu - 4}{\nu - 1} \int \frac{\partial \Phi}{\sin. \Phi^{\mu - 2} \cos. \Phi^{\nu - 2}},$$

vnde concluditur

$$\int \frac{\partial \Phi}{\sin. \Phi^{\mu} \cos. \Phi^{\nu}} = -\frac{1}{\mu - 1} \cdot \frac{1}{\sin. \Phi^{\mu - 1} \cos. \Phi^{\nu - 1}}$$
$$+ \frac{\mu + \nu - 2}{(\mu - 1)(\nu - 1)} \cdot \frac{1}{\sin. \Phi^{\mu - 3} \cos. \Phi^{\nu - 1}}$$
$$+ \frac{(\mu + \nu - 2)(\mu + \nu - 4)}{(\mu - 1)(\nu - 1)} \int \frac{\partial \Phi}{\sin. \Phi^{\mu - 2} \cos. \Phi^{\nu - 2}}$$

Corollarium 2.

259. Prioribus membris ad communem denominatorem reductis obtinebitur

$$\int \frac{\partial \Phi}{\sin. \Phi^{\mu} \cos. \Phi^{\nu}} = \frac{(\mu - 1) \sin. \Phi^{2} - (\nu - 1) \cos. \Phi^{2}}{(\mu - 1)(\nu - 1) \sin. \Phi^{\mu - 1} \cos. \Phi^{\nu - 1}}$$
$$+ \frac{(\mu + \nu - 2)(\mu + \nu - 4)}{(\mu - 1)(\nu - 1)} \int \frac{\partial \Phi}{\sin. \Phi^{\mu - 2} \cos. \Phi^{\nu - 2}}$$

qua reductione semper ad calculum contrahendum vti licet, nisi vel $\mu = 1$ vel $\nu = 1$.

Scho-

Scholion.

260. Huiusmodi formulae $\frac{\partial\Phi}{\text{fin.}\,\Phi^m\,\text{cof.}\,\Phi^n}$ etiam hoc modo maxime obuio ad fimpliciores reduci poffunt; dum numerator per $\text{fin.}\,\Phi^2 + \text{cof.}\,\Phi^2 = 1$ multiplicatur, vnde fit

$$\int\frac{\partial\Phi}{\text{fin.}\,\Phi^m\,\text{cof.}\,\Phi^n} = \int\frac{\partial\Phi}{\text{fin.}\,\Phi^{m-2}\,\text{cof.}\,\Phi^n} + \int\frac{\partial\Phi}{\text{fin.}\,\Phi^m\,\text{cof.}\,\Phi^{n-1}}$$

quae eousque continuari poteft, donec in denominatore vnica tantum poteftas relinquatur. Ita erit

$$\int\frac{\partial\Phi}{\text{fin.}\,\Phi\,\text{cof.}\,\Phi} = \int\frac{\partial\Phi\,\text{fin.}\,\Phi}{\text{cof.}\,\Phi} + \int\frac{\partial\Phi\,\text{cof.}\,\Phi}{\text{fin.}\,\Phi} = l\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi}$$

$$\int\frac{\partial\Phi}{\text{fin.}\,\Phi^2\,\text{cof.}\,\Phi^2} = \int\frac{\partial\Phi}{\text{fin.}\,\Phi^2} + \int\frac{\partial\Phi}{\text{cof.}\,\Phi^2} = \frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi} - \frac{\text{cof.}\,\Phi}{\text{fin.}\,\Phi}.$$

Quodfi propofita fit haec formula $\int\frac{\partial\Phi}{\text{fin.}\,\Phi^n\,\text{cof.}\,\Phi^n}$, in fubfidium vocari poteft, effe $\text{fin.}\,\Phi\,\text{cof.}\,\Phi = \frac{1}{2}\,\text{fin.}\,2\Phi$, vnde habetur $\int\frac{2^n\,\partial\Phi}{\text{fin.}\,2\Phi^n} = 2^{n-1}\int\frac{\partial\omega}{\text{fin.}\,\omega^n}$, pofito $\omega = 2\Phi$, quae formula per fuperiora praecepta refoluitur. His igitur adminiculis obferuatis circa formulam $\partial\Phi\,\text{fin.}\,\Phi^m\,\text{cof.}\,\Phi^n$, fi quidem m et n fuerint numeri integri fiue pofitiui fiue negatiui, nihil amplius defideratur: fin autem fuerint numeri fracti, nihil admodum praecipiendum occurrit, quandoquidem cafus, quibus integratio fuccedit, quafi fponte fe produnt. Quemadmodum autem integralia, quae exhiberi nequeunt, per feries exprimi conueniat, in capite fequente accuratius exponamus. Nunc vero formulas fractas confideremus, quarum denominator eft $a + b\,\text{cof.}\,\Phi$ eiusque poteftas, tales enim formulae in Theoria Aftronomiae frequentiffime occurrunt.

Problema 29.

261. Formulae differentialis $\frac{\partial\Phi}{a + b\,\text{cof.}\,\Phi}$ integrale inueftigare.

Solutio.

Haec inueſtigatio commodius inſtitui nequit, quam vt formula propoſita ad formam ordinariam reducatur, ponendo $\text{cof.}\,\Phi = \frac{1-xx}{1+xx}$, vt rationaliter fiat $\text{fin.}\,\Phi = \frac{2x}{1+xx}$, hincque $\partial\Phi\,\text{cof.}\,\Phi = \frac{2\partial x(1-xx)}{(1+xx)^2}$, ficque $\partial\Phi = \frac{2\partial x}{1+xx}$. Quia igitur $a+b\,\text{cof.}\,\Phi = \frac{a+b+(a-b)xx}{1+xx}$, erit formula noſtra $\frac{\partial\Phi}{a+b\,\text{cof.}\,\Phi} = \frac{2\partial x}{a+b+(a-b)xx}$, quae prout fuerit $a > b$ vel $a < b$, vel angulum vel logarithmum praebet.

Caſu $a > b$ reperitur

$$\int \frac{\partial\Phi}{a+b\,\text{cof.}\,\Phi} = \frac{2}{\sqrt{(aa-bb)}}\,\text{Arc. tang.}\,\frac{(a-b)x}{\sqrt{(aa-bb)}}:$$

caſu $a < b$ vero eſt

$$\int \frac{\partial\Phi}{a+b\,\text{cof.}\,\Phi} = \frac{1}{\sqrt{(bb-aa)}}\,l\,\frac{\sqrt{(bb-aa)}+x(b-a)}{\sqrt{(bb-aa)}-x(b-a)}.$$

Nunc vero eſt

$$x = \sqrt{\frac{1-\text{cof.}\,\Phi}{1+\text{cof.}\,\Phi}} = \text{tang.}\,\tfrac{1}{2}\Phi = \frac{\text{fin.}\,\Phi}{1+\text{cof.}\,\Phi};$$

qua reſtitutione facta, cum ſit

$$2\,\text{Ang. tang.}\,\frac{(a-b)x}{\sqrt{(aa-bb)}} = \text{Ang. tang.}\,\frac{2x\sqrt{(aa-bb)}}{a+b-(a-b)xx}$$

$$= \text{Ang. tang.}\,\frac{2\,\text{fin.}\,\Phi\sqrt{(aa-bb)}}{(a+b)(1+\text{cof.}\,\Phi)-(a-b)(1-\text{cof.}\,\Phi)}$$

$$= \text{Ang. tang.}\,\frac{\text{fin.}\,\Phi\sqrt{(aa-bb)}}{a\,\text{cof.}\,\Phi+b}.$$

Quocirca pro caſu $a > b$ adipiſcimur:

$$\int \frac{\partial\Phi}{a+b\,\text{cof.}\,\Phi} = \frac{1}{\sqrt{(aa-bb)}}\,\text{Ang. tang.}\,\frac{\text{fin.}\,\Phi.\sqrt{(aa-bb)}}{a\,\text{cof.}\,\Phi+b},\ \text{feu}$$

$$\int \frac{\partial\Phi}{a+b\,\text{cof.}\,\Phi} = \frac{1}{\sqrt{(aa-bb)}}\,\text{Ang. fin.}\,\frac{\text{fin.}\,\Phi.\sqrt{(aa-bb)}}{a+b\,\text{cof.}\,\Phi},\ \text{fiue}$$

$$\int \frac{\partial\Phi}{a+b\,\text{cof.}\,\Phi} = \frac{1}{\sqrt{(aa-bb)}}\,\text{Ang. cof.}\,\frac{a\,\text{cof.}\,\Phi+b}{a+b\,\text{cof.}\,\Phi}.$$

Pro caſu autem $a < b$:

$$\int \frac{\partial\Phi}{a+b\,\text{cof.}\,\Phi} = \frac{1}{\sqrt{(bb-aa)}}\,l\,\frac{\sqrt{(b+a)(1+\text{cof.}\,\Phi)}+\sqrt{(b-a)(1-\text{cof.}\,\Phi)}}{\sqrt{(b+a)(1+\text{cof.}\,\Phi)}-\sqrt{(b-a)(1-\text{cof.}\,\Phi)}},$$

ſeu

$$\int \frac{\partial\Phi}{a+b\,\text{cof.}\,\Phi} = \frac{1}{\sqrt{(bb-aa)}}\,l\,\frac{a\,\text{cof.}\,\Phi+b+\text{fin.}\,\Phi.\sqrt{(bb-aa)}}{a+b\,\text{cof.}\,\Phi}.$$

At

At casu $b = a$, integrale est $= \frac{x}{a} = \frac{1}{2}$ tang. $\frac{1}{2}\Phi$, vnde fit

$$\int \frac{\partial\Phi}{1 + \cos.\Phi} = \text{tang.}\ \tfrac{1}{2}\Phi = \frac{\sin.\Phi}{1 + \cos.\Phi},$$

quae integralia euanescunt facto $\Phi = 0$.

Corollarium 1.

262. Formulae autem $\frac{\partial\Phi \sin.\Phi}{a + b\cos.\Phi} = \frac{-\partial.\cos.\Phi}{a + b\cos.\Phi}$ integrale est $\frac{1}{b} l \frac{a + b}{a + b\cos.\Phi}$, ita sumtum, vt euanescat posito $\Phi = 0$; sicque habebimus:

$$\int \frac{\partial\Phi \sin.\Phi}{a + b\cos.\Phi} = \frac{1}{b} l \frac{a + b}{a + b\cos.\Phi}.$$

Corollarium 2.

263. Formula autem $\frac{\partial\Phi \cos.\Phi}{a + b\cos.\Phi}$ transformatur in $\frac{\partial\Phi}{b} - \frac{a\,\partial\Phi}{b(a + b\cos.\Phi)}$, vnde integrale per solutionem problematis exhiberi potest:

$$\int \frac{\partial\Phi \cos.\Phi}{a + b\cos.\Phi} = \frac{\Phi}{b} - \frac{a}{b}\int \frac{\partial\Phi}{a + b\cos.\Phi}.$$

Scholion 1.

264. Integratione hac inuenta, etiam huius formulae $\frac{\partial\Phi}{(a + b\cos.\Phi)^n}$ integrale inueniri potest, existente n numero integro; quod fingendo integralis forma commodissime praestari videtur: ponatur

$$\int \frac{\partial\Phi}{(a + b\cos.\Phi)^2} = \frac{A\sin.\Phi}{a + b\cos.\Phi} + m\int \frac{\partial\Phi}{a + b\cos.\Phi};$$

ac reperitur

$$A = \frac{-b}{aa - bb}, \text{ et } m = \frac{a}{aa + bb}.$$ Porro fingatur

$$\int \frac{\partial\Phi}{(a + b\cos.\Phi)^3} = \frac{(A + B\cos.\Phi)\sin.\Phi}{(a + b\cos.\Phi)^2} + m\int \frac{\partial\Phi}{(a + b\cos.\Phi)^2};$$

reperiturque

$$A = \frac{-b}{aa - bb};\ B = \frac{-ab}{2a(aa - bb)};\ m = \frac{2aa + bb}{2a(aa - bb)};$$

 simi-

ſimilique modo inueſtigatio ad maiores poteſtates continuari poteſt, labore quidem non parum taedioſo. Sequenti autem modo negotium facillime expediri videtur.

Conſideretur ſcilicet formula generalior $\frac{\partial\Phi(f+g\cos.\Phi)}{(a+b\cos.\Phi)^{n+1}}$, ac ponatur

$$\int\frac{\partial\Phi(f+g\cos.\Phi)}{(a+b\cos.\Phi)^{n+1}}=\frac{A\sin.\Phi}{(a+b\cos.\Phi)^{n}}+\int\frac{\partial\Phi(B+C\cos.\Phi)}{(a+b\cos.\Phi)^{n}},$$

ſumtisque differentialibus, iſta prodibit aequatio:

$$\begin{aligned} f+g\cos.\Phi = {} & A\cos.\Phi(a+b\cos.\Phi)+nAb\sin.\Phi^2 \\ & +(B+C\cos.\Phi)(a+b\cos.\Phi); \end{aligned}$$

quae ob $\sin.\Phi^2=1-\cos.\Phi^2$ hanc formam induit

$$\left.\begin{matrix} -f & -g\cos.\Phi & +Ab\cos.\Phi^2 \\ +nAb & +Aa\cos.\Phi & -nAb\cos.\Phi^2 \\ +Ba & +Bb\cos.\Phi & +Cb\cos.\Phi^2 \\ & +Ca\cos.\Phi & \end{matrix}\right\}=0;$$

vnde ſingulis membris nihilo aequatis, elicitur:

$$A=\frac{ag-bf}{n(aa-bb)};\quad B=\frac{af-bg}{aa-bb}\quad\text{et}\quad C=\frac{(n-1)(ag-bf)}{n(aa-bb)}.$$

Ita vt haec obtineatur reductio

$$\int\frac{\partial\Phi(f+g\cos.\Phi)}{(a+b\cos.\Phi)^{n+1}}=\frac{(ag-bf)\sin.\Phi}{n(aa-bb)(a+b\cos.\Phi)^n}$$
$$+\frac{1}{n(aa-bb)}\int\frac{\partial\Phi[n(af-bg)+(n-1)(ag-bf)\cos.\Phi]}{(a+b\cos.\Phi)^n}$$

cuius ope tandem ad formulam $\int\frac{\partial\Phi(b+k\cos.\Phi)}{a+b\cos.\Phi}$ peruenitur, cuius integrale $=\frac{k}{b}\Phi+\frac{bb-ak}{b}\int\frac{\partial\Phi}{a+b\cos.\Phi}$ ex ſuperioribus conſtat. Perſpicuum autem eſt ſemper fore $k=0$.

A

Scho-

Scholion 2.

265. Occurrunt etiam eiusmodi formulae, in quas insuper quantitas exponentialis $e^{\alpha\Phi}$, angulum ipsum Φ in exponente gerens, ingreditur, quas quomodo tractari oporteat, ostendendum videtur, cum hinc methodus reductionum supra exposita maxime illustretur. Hic enim per illam reductionem ad formulam propositae similem peruenitur, vnde ipsum integrale colligi poterit. In hunc finem notetur esse $\int e^{\alpha\Phi}\partial\Phi = \frac{1}{\alpha}e^{\alpha\Phi}$.

Problema 30.

266. Formulae differentialis $\partial y = e^{\alpha\Phi}\partial\Phi \,\text{fin.}\,\Phi^n$ integrale inuestigare.

Solutio.

Sumto $e^{\alpha\Phi}\partial\Phi$ pro factore differentiali, erit

$$y = \tfrac{1}{\alpha}e^{\alpha\Phi}\,\text{fin.}\,\Phi^n - \tfrac{n}{\alpha}\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^{n-1}\,\text{cof.}\,\Phi:$$

fimili modo reperitur

$$\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^{n-1}\,\text{cof.}\,\Phi = \tfrac{1}{\alpha}e^{\alpha\Phi}\,\text{fin.}\,\Phi^{n-1}\,\text{cof.}\,\Phi$$
$$-\tfrac{1}{\alpha}\int e^{\alpha\Phi}\partial\Phi\,[(n-1)\,\text{fin.}\,\Phi^{n-2}\,\text{cof.}\,\Phi^2 - \text{fin.}\,\Phi^n],$$

quae postrema formula, ob $\text{cof.}\,\Phi^2 = 1 - \text{fin.}\,\Phi^2$, reducitur ad has

$$(n-1)\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^{n-2} - n\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^n:$$

vnde habebitur

$$\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^n = \tfrac{1}{\alpha}e^{\alpha\Phi}\,\text{fin.}\,\Phi^n - \tfrac{n}{\alpha\alpha}e^{\alpha\Phi}\,\text{fin.}\,\Phi^{n-1}\,\text{cof.}\,\Phi$$
$$+\tfrac{n(n-1)}{\alpha\alpha}\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^{n-2} - \tfrac{nn}{\alpha\alpha}\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^n.$$

Quare hanc postremam formulam cum prima coniungendo, elicitur:

$$\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^n = \frac{e^{\alpha\Phi}\,\text{fin.}\,\Phi^{n-1}(\alpha\,\text{fin.}\,\Phi - n\,\text{cof.}\,\Phi)}{\alpha\alpha + nn}$$
$$+\frac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^{n-2}.$$

Duo-

Duobus ergo casibus integrale absolute datur, scilicet $n = 0$ et $n = 1$, eritque

$$\int e^{\alpha\Phi}\partial\Phi = \tfrac{1}{\alpha}e^{\alpha\Phi} - \tfrac{1}{\alpha}, \text{ et}$$

$$\int e^{\alpha\Phi}\partial\Phi \,\text{fin.}\,\Phi = \frac{e^{\alpha\Phi}(\alpha\,\text{fin.}\,\Phi - \text{cof.}\,\Phi)}{\alpha\alpha+1} + \frac{1}{\alpha\alpha+1}$$

atque ad hos sequentes omnes, vbi n est numerus integer vnitate maior, reducuntur.

Corollarium 1.

267. Ita si $n = 2$, acquirimus hanc integrationem

$$\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^2 = \frac{e^{\alpha\Phi}\,\text{fin.}\,\Phi\,(\alpha\,\text{fin.}\,\Phi - 2\,\text{cof.}\,\Phi)}{\alpha\alpha+4}$$

$$+ \frac{1.2}{\alpha(\alpha\alpha+4)}e^{\alpha\Phi} - \frac{1.2}{\alpha(\alpha\alpha+4)}:$$

at si sit $n = 3$, istam

$$\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^3 = \frac{e^{\alpha\Phi}\,\text{fin.}\,\Phi^2\,(\alpha\,\text{fin.}\,\Phi - 3\,\text{cof.}\,\Phi)}{\alpha\alpha+9}$$

$$+ \frac{2.3\,e^{\alpha\Phi}(\alpha\,\text{fin.}\,\Phi - \text{cof.}\,\Phi)}{(\alpha\alpha+1)(\alpha\alpha+9)} + \frac{2.3}{(\alpha\alpha+1)(\alpha\alpha+9)}.$$

Integralibus ita sumtis, vt euanescant, posito $\Phi = 0$.

Corollarium 2.

268. Si igitur determinatis hoc modo integralibus, statuatur $\alpha\Phi = -\infty$, vt $e^{\alpha\Phi}$ euanescat, erit in genere

$$\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^n = \tfrac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^{n-2};$$

hincque integralia pro isto casu $\alpha\Phi = -\infty$ erunt

$$\int e^{\alpha\Phi}\partial\Phi = -\tfrac{1}{\alpha}; \qquad \int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi = \tfrac{1}{\alpha\alpha+1};$$

$$\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^2 = \tfrac{-1.2}{\alpha(\alpha\alpha+4)}; \qquad \int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^3 = \tfrac{1.2.3}{(\alpha\alpha+1)(\alpha\alpha+9)};$$

$$\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^4 = \tfrac{-1.2.3.4}{\alpha(\alpha\alpha+4)(\alpha\alpha+16)}; \qquad \int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^5 = \tfrac{1.2.3.4.5}{(\alpha\alpha+1)(\alpha\alpha+9)(\alpha\alpha+25)}.$$

Corol-

Corollarium 3.

269. Quare si proponatur haec series infinita

$$s = 1 + \frac{1.2}{\alpha\alpha+4} + \frac{1.2.3.4}{(\alpha\alpha+4)(\alpha\alpha+16)} + \frac{1.2.3.4.5.6}{(\alpha\alpha+4)(\alpha\alpha+16)(\alpha\alpha+36)} + \text{etc. erit}$$

$$s = -\alpha \int e^{\alpha\Phi} \partial\Phi (1 + \text{fin.}\,\Phi^2 + \text{fin.}\,\Phi^4 + \text{fin.}\,\Phi^6 + \text{etc.})$$

seu $s = -\alpha \int \frac{e^{\alpha\Phi}\,\partial\Phi}{\text{cof.}\,\Phi^2}$, posito post integrationem $\alpha\Phi = -\infty$.

Problema 31.

270. Formulae differentialis $e^{\alpha\Phi}\,\partial\Phi\,\text{cof.}\,\Phi^n$ integrale inuestigare.

Solutio.

Simili modo procedendo vt ante, erit

$$e^{\alpha\Phi}\partial\Phi\,\text{cof.}\,\Phi^n = \tfrac{1}{\alpha} e^{\alpha\Phi}\text{cof.}\,\Phi^n + \tfrac{n}{\alpha}\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi\,\text{cof.}\,\Phi^{n-1}:$$

tum vero

$$\int e^{\alpha\Phi}\,\partial\Phi\,\text{fin.}\,\Phi\,\text{cof.}\,\Phi^{n-1} = \tfrac{1}{\alpha} e^{\alpha\Phi}\,\text{fin.}\,\Phi\,\text{cof.}\,\Phi^{n-1}$$
$$- \tfrac{1}{\alpha}\int e^{\alpha\Phi}\,\partial\Phi\,[\text{cof.}\,\Phi^n - (n-1)\,\text{cof.}\,\Phi^{n-2}\,\text{fin.}\,\Phi^2],$$

quae postrema formula abit in $-(n-1)\int e^{\alpha\Phi}\,\partial\Phi\,\text{cof.}\,\Phi^{n-2} + n\int e^{\alpha\Phi}\,\partial\Phi\,\text{cof.}\,\Phi^n$, ita vt sit

$$\int e^{\alpha\Phi}\,\partial\Phi\,\text{cof.}\,\Phi^n = \tfrac{1}{\alpha} e^{\alpha\Phi}\,\text{cof.}\,\Phi^n + \tfrac{n}{\alpha\alpha} e^{\alpha\Phi}\,\text{fin.}\,\Phi\,\text{cof.}\,\Phi^{n-1}$$
$$+ \tfrac{n(n-1)}{\alpha\alpha}\int e^{\alpha\Phi}\,\partial\Phi\,\text{cof.}\,\Phi^{n-2} - \tfrac{nn}{\alpha\alpha}\int e^{\alpha\Phi}\,\partial\Phi\,\text{cof.}\,\Phi^n,$$

vnde colligimus

$$\int e^{\alpha\Phi}\,\partial\Phi\,\text{cof.}\,\Phi^n = \frac{e^{\alpha\Phi}\,\text{cof.}\,\Phi^{n-1}\,(\alpha\,\text{cof.}\,\Phi + n\,\text{fin.}\,\Phi)}{\alpha\alpha + nn}$$
$$+ \frac{n(n-1)}{\alpha\alpha + nn}\int e^{\alpha\Phi}\,\partial\Phi\,\text{cof.}\,\Phi^{n-2}.$$

Hinc ergo casus simplicissimi sunt

$$\int e^{\alpha\Phi}\partial\Phi = \tfrac{1}{\alpha} e^{\alpha\Phi} + C;\quad \int e^{\alpha\Phi}\partial\Phi\,\text{cof.}\,\Phi = \frac{e^{\alpha\Phi}(\alpha\,\text{cof.}\,\Phi + \text{fin.}\,\Phi)}{\alpha\alpha+1} + C,$$

ad quos sequentes omnes, vbi n est numerus integer positiuus, reducuntur.

Scholion.

271. Casibus simplicissimis notatis, alia datur via integrale formularum propositarum, quin etiam huius magis patentis $e^{\alpha\Phi}\partial\Phi \text{ fin. } \Phi^m \text{ cof. } \Phi^n$, eruendi. Cum enim productum $\text{fin. } \Phi^m \text{ cof. } \Phi^n$ resolui possit in aggregatum plurium sinuum vel cosinuum, quorum quisque est huius formae $M \text{ fin. } \lambda\Phi$ vel $M \text{ cof. } \lambda\Phi$, integratio reducitur ad alterutram harum formularum $e^{\alpha\Phi}\partial\Phi \text{ fin. } \lambda\Phi$ vel $e^{\alpha\Phi}\partial\Phi \text{ cof. } \lambda\Phi$. Ponamus ergo $\lambda\Phi = \omega$, vt habemus

$$e^{\alpha\Phi}\partial\Phi \text{ cof. } \lambda\Phi = \tfrac{1}{\lambda} e^{\frac{\alpha}{\lambda}\omega}\partial\omega \text{ fin. } \omega, \text{ et}$$

$$e^{\alpha\Phi}\partial\Phi \text{ cof. } \lambda\Phi = \tfrac{1}{\lambda} e^{\frac{\alpha}{\lambda}\omega}\partial\omega \text{ cof. } \omega,$$

quarum integralia per superiora ita dantur:

$$\int e^{\frac{\alpha}{\lambda}\omega}\partial\omega \text{ fin. } \omega = \frac{\lambda e^{\frac{\alpha}{\lambda}\omega}(\alpha \text{ fin. } \omega - \lambda \text{ cof. } \omega)}{\alpha\alpha+\lambda\lambda} = \frac{\lambda e^{\alpha\Phi}(\alpha \text{ fin. } \lambda\Phi - \lambda \text{ cof. } \lambda\Phi)}{\alpha\alpha+\lambda\lambda},$$

$$\int e^{\frac{\alpha}{\lambda}\omega}\partial\omega \text{ cof. } \omega = \frac{\lambda e^{\frac{\alpha}{\lambda}\omega}(\alpha \text{ cof. } \omega + \lambda \text{ fin. } \omega)}{\alpha\alpha+\lambda\lambda} = \frac{\lambda e^{\alpha\Phi}(\alpha \text{ cof. } \lambda\Phi + \lambda \text{ fin. } \lambda\Phi)}{\alpha\alpha+\lambda\lambda}.$$

Vnde tandem colligimus:

$$\int e^{\alpha\Phi}\partial\Phi \text{ fin. } \lambda\Phi = \frac{e^{\alpha\Phi}(\alpha \text{ fin. } \lambda\Phi - \lambda \text{ cof. } \lambda\Phi)}{\alpha\alpha+\lambda\lambda}, \text{ et}$$

$$\int e^{\alpha\Phi}\partial\Phi \text{ cof. } \lambda\Phi = \frac{e^{\alpha\Phi}(\alpha \text{ cof. } \lambda\Phi + \lambda \text{ fin. } \lambda\Phi)}{\alpha\alpha+\lambda\lambda},$$

si in genere statim loco $\text{fin. } \Phi$ et $\text{cof. } \Phi$ scripsissem $\text{fin. } \lambda\Phi$ et $\text{cof. } \lambda\Phi$, hac reductione non fuisset opus; sed quia hic nihil est difficultatis, breuitati consulendum existimaui.

CA-

CAPVT VI.

DE EVOLVTIONE INTEGRALIVM PER SERIES, SECVNDVM SINVS COSINVSVE ANGVLORVM MVLTIPLORVM PROGREDIENTES.

Problema 32.

272.

Integrale formulae $\frac{\partial\Phi}{1+n\,\mathrm{cos}.\Phi}$ per feriem, fecundum finus angulorum multiplorum progredientem, exprimere.

Solutio

Cum fit more confueto per feriem

$$\frac{1}{1+n\,\mathrm{cos}.\Phi} = 1 - n\,\mathrm{cos}.\Phi + n^2\,\mathrm{cos}.\Phi^2 - n^3\,\mathrm{cos}.\Phi^3 + n^4\,\mathrm{cos}.\Phi^4 - \text{etc.}$$

poteftates cofinus in cofinus angulorum multiplorum conuertantur ope formularum in introductione traditarum: ac primo pro poteftatibus imparibus:

$$\mathrm{cof}.\Phi = \mathrm{cof}.\Phi;$$

$$\mathrm{cof}.\Phi^3 = \tfrac{3}{4}\,\mathrm{cof}.\Phi + \tfrac{1}{4}\,\mathrm{cof}.3\Phi;$$

$$\mathrm{cof}.\Phi^5 = \tfrac{10}{16}\,\mathrm{cof}.\Phi + \tfrac{5}{16}\,\mathrm{cof}.3\Phi + \tfrac{1}{16}\,\mathrm{cof}.5\Phi;$$

$$\mathrm{cof}.\Phi^7 = \tfrac{35}{64}\,\mathrm{cof}.\Phi + \tfrac{21}{64}\,\mathrm{cof}.3\Phi + \tfrac{7}{64}\,\mathrm{cof}.5\Phi + \tfrac{1}{64}\,\mathrm{cof}.7\Phi;$$

$$\mathrm{cof}.\Phi^9 = \tfrac{126}{256}\,\mathrm{cof}.\Phi + \tfrac{84}{256}\,\mathrm{cof}.3\Phi + \tfrac{36}{256}\,\mathrm{cof}.5\Phi + \tfrac{9}{256}\,\mathrm{cof}.7\Phi + \tfrac{1}{256}\,\mathrm{cof}.9\Phi;$$

vbi notandum eft, fi ponatur in genere

$$\mathrm{cof}.\Phi^{2\lambda+1} = A\,\mathrm{cof}.\Phi + B\,\mathrm{cof}.3\Phi + C\,\mathrm{cof}.5\Phi + D\,\mathrm{cof}.7\Phi + E\,\mathrm{cof}.9\Phi + \text{etc.}$$

ad quos ſequentes omnes, vbi n eſt numerus integer poſitiuus, reducuntur.

Scholion.

271. Caſibus ſimpliciſſimis notatis, alia datur via integrale formularum propoſitarum, quin etiam huius magis patentis $e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\Phi^m\,\text{cof.}\,\Phi^n$, eruendi. Cum enim productum $\text{fin.}\,\Phi^m\,\text{cof.}\,\Phi^n$ reſolui poſſit in aggregatum plurium ſinuum vel coſinuum, quorum quisque eſt huius formae $M\,\text{fin.}\,\lambda\Phi$ vel $M\,\text{cof.}\,\lambda\Phi$, integratio reducitur ad alterutram harum formularum $e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\lambda\Phi$ vel $e^{\alpha\Phi}\partial\Phi\,\text{cof.}\,\lambda\Phi$. Ponamus ergo $\lambda\Phi = \omega$, vt habemus

$$e^{\alpha\Phi}\partial\Phi\,\text{cof.}\,\lambda\Phi = \tfrac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}\partial\omega\,\text{fin.}\,\omega, \text{ et}$$

$$e^{\alpha\Phi}\partial\Phi\,\text{cof.}\,\lambda\Phi = \tfrac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}\partial\omega\,\text{cof.}\,\omega,$$

quarum integralia per ſuperiora ita dantur:

$$\int e^{\frac{\alpha}{\lambda}\omega}\partial\omega\,\text{fin.}\,\omega = \frac{\lambda e^{\frac{\alpha}{\lambda}\omega}(\alpha\,\text{fin.}\,\omega - \lambda\,\text{cof.}\,\omega)}{\alpha\alpha+\lambda\lambda} = \frac{\lambda e^{\alpha\Phi}(\alpha\,\text{fin.}\,\lambda\Phi - \lambda\,\text{cof.}\,\lambda\Phi)}{\alpha\alpha+\lambda\lambda},$$

$$\int e^{\frac{\alpha}{\lambda}\omega}\partial\omega\,\text{cof.}\,\omega = \frac{\lambda e^{\frac{\alpha}{\lambda}\omega}(\alpha\,\text{cof.}\,\omega + \lambda\,\text{fin.}\,\omega)}{\alpha\alpha+\lambda\lambda} = \frac{\lambda e^{\alpha\Phi}(\alpha\,\text{cof.}\,\lambda\Phi + \lambda\,\text{fin.}\,\lambda\Phi)}{\alpha\alpha+\lambda\lambda}.$$

Vnde tandem colligimus:

$$\int e^{\alpha\Phi}\partial\Phi\,\text{fin.}\,\lambda\Phi = \frac{e^{\alpha\Phi}(\alpha\,\text{fin.}\,\lambda\Phi - \lambda\,\text{cof.}\,\lambda\Phi)}{\alpha\alpha+\lambda\lambda}, \text{ et}$$

$$\int e^{\alpha\Phi}\partial\Phi\,\text{cof.}\,\lambda\Phi = \frac{e^{\alpha\Phi}(\alpha\,\text{cof.}\,\lambda\Phi + \lambda\,\text{fin.}\,\lambda\Phi)}{\alpha\alpha+\lambda\lambda},$$

ſi in genere ſtatim loco $\text{fin.}\,\Phi$ et $\text{cof.}\,\Phi$ ſcripſiſſem $\text{fin.}\,\lambda\Phi$ et $\text{cof.}\,\lambda\Phi$, hac reductione non fuiſſet opus; ſed quia hic nihil eſt difficultatis, breuitati conſulendum exiſtimaui.

CA-

CAPVT VI.

DE EVOLVTIONE INTEGRALIVM PER SERIES, SECVNDVM SINVS COSINVSVE ANGVLORVM MVLTIPLORVM PROGREDIENTES.

Problema 32.

272.

Integrale formulae $\frac{\partial \Phi}{1 + n \operatorname{cof.} \Phi}$ per feriem, fecundum finus angulorum multiplorum progredientem, exprimere.

Solutio

Cum fit more confueto per feriem

$$\frac{1}{1 + n \operatorname{cof.} \Phi} = 1 - n \operatorname{cof.} \Phi + n^2 \operatorname{cof.} \Phi^2 - n^3 \operatorname{cof.} \Phi^3 + n^4 \operatorname{cof.} \Phi^4 - \text{etc.}$$

poteftates cofinus in cofinus angulorum multiplorum conuertantur ope formularum in introductione traditarum: ac primo pro poteftatibus imparibus:

$$\operatorname{cof.} \Phi = \operatorname{cof.} \Phi;$$

$$\operatorname{cof.} \Phi^3 = \tfrac{3}{4} \operatorname{cof.} \Phi + \tfrac{1}{4} \operatorname{cof.} 3\Phi;$$

$$\operatorname{cof.} \Phi^5 = \tfrac{10}{16} \operatorname{cof.} \Phi + \tfrac{5}{16} \operatorname{cof.} 3\Phi + \tfrac{1}{16} \operatorname{cof.} 5\Phi;$$

$$\operatorname{cof.} \Phi^7 = \tfrac{35}{64} \operatorname{cof.} \Phi + \tfrac{21}{64} \operatorname{cof.} 3\Phi + \tfrac{7}{64} \operatorname{cof.} 5\Phi + \tfrac{1}{64} \operatorname{cof.} 7\Phi;$$

$$\operatorname{cof.} \Phi^9 = \tfrac{126}{256} \operatorname{cof.} \Phi + \tfrac{84}{256} \operatorname{cof.} 3\Phi + \tfrac{36}{256} \operatorname{cof.} 5\Phi + \tfrac{9}{256} \operatorname{cof.} 7\Phi + \tfrac{1}{256} \operatorname{cof.} 9\Phi;$$

vbi notandum eft, fi ponatur in genere

$$\operatorname{cof.} \Phi^{2\lambda - 1} = A \operatorname{cof.} \Phi + B \operatorname{cof.} 3\Phi + C \operatorname{cof.} 5\Phi + D \operatorname{cof.} 7\Phi + E \operatorname{cof.} 9\Phi + \text{etc.}$$

 fore

fore

$$A = 2 . \frac{1 . 3 . 5 . \dots \dots (2\lambda - 1)}{2 . 4 . 6 . \dots \dots \quad 2\lambda} = \frac{2}{2^{2\lambda-1}} . \frac{6}{2} . \frac{10}{3} . \frac{14}{4} \dots . \frac{4\lambda - 2}{\lambda};$$

$$B = \tfrac{\lambda-1}{\lambda+1} A; \; C = \tfrac{\lambda-2}{\lambda+2} B; \; D = \tfrac{\lambda-3}{\lambda+3} C; \; E = \tfrac{\lambda-4}{\lambda+4} D; \text{ etc.}$$

Pro paribus vero poteſtatibus eſt

$$\begin{aligned}
&\text{coſ.}\,\Phi^0 = 1 \\
&\text{coſ.}\,\Phi^2 = \tfrac{1}{2} + \tfrac{1}{2}\,\text{coſ.}\,2\Phi \\
&\text{coſ.}\,\Phi^4 = \tfrac{3}{8} + \tfrac{4}{8}\,\text{coſ.}\,2\Phi + \tfrac{1}{8}\,\text{coſ.}\,4\Phi \\
&\text{coſ.}\,\Phi^6 = \tfrac{10}{32} + \tfrac{15}{32}\,\text{coſ.}\,2\Phi + \tfrac{6}{32}\,\text{coſ.}\,4\Phi + \tfrac{1}{32}\,\text{coſ.}\,6\Phi \\
&\text{coſ.}\,\Phi^8 = \tfrac{35}{128} + \tfrac{56}{128}\,\text{coſ.}\,2\Phi + \tfrac{28}{128}\,\text{coſ.}\,4\Phi + \tfrac{8}{128}\,\text{coſ.}\,6\Phi + \tfrac{1}{128}\,\text{coſ.}\,8\Phi.
\end{aligned}$$

In genere autem ſi ponatur:

$$\text{coſ.}\,\Phi^{2\lambda} = \mathfrak{A} + \mathfrak{B}\,\text{coſ.}\,2\Phi + \mathfrak{C}\,\text{coſ.}\,4\Phi + \mathfrak{D}\,\text{coſ.}\,6\Phi$$
$$+ \mathfrak{E}\,\text{coſ.}\,8\Phi + \text{etc.} \quad \text{erit}$$

$$\mathfrak{A} = \frac{1 . 3 . 5 . \dots \dots (2\lambda - 1)}{2 . 4 . 6 . \dots \dots \quad 2\lambda} = \frac{1}{2^{2\lambda-1}} . \frac{6}{2} . \frac{10}{3} . \frac{14}{4} \dots . \frac{4\lambda - 2}{\lambda}$$

$$\mathfrak{B} = \tfrac{2\lambda}{\lambda+1}\mathfrak{A}; \; \mathfrak{C} = \tfrac{\lambda-1}{\lambda+2}\mathfrak{B}; \; \mathfrak{D} = \tfrac{\lambda-2}{\lambda+3}\mathfrak{C}; \; \mathfrak{E} = \tfrac{\lambda-3}{\lambda+4}\mathfrak{D}; \text{ etc.}$$

Quodſi nunc iſti valores ſubſtituantur, erit $\frac{1}{1 + n\,\text{coſ.}\,\Phi} =$

$$\begin{array}{lllllll}
1 & -n\,\text{coſ.}\Phi & +\tfrac{1}{2}nn\,\text{coſ.}2\Phi & -\tfrac{1}{4}n^3\,\text{coſ.}3\Phi & +\tfrac{1}{8}n^4\,\text{coſ.}4\Phi & -\tfrac{1}{16}n^5\,\text{coſ.}5\Phi & +\tfrac{1}{32}n^6\,\text{coſ.}6\Phi \\
+\tfrac{1}{2}nn & -\tfrac{3}{4}n^3 & +\tfrac{4}{8}n^4 & -\tfrac{5}{16}n^5 & +\tfrac{6}{32}n^6 & -\tfrac{7}{64}n^7 & +\tfrac{8}{128}n^8 \\
+\tfrac{3}{8}n^4 & -\tfrac{10}{16}n^5 & +\tfrac{15}{32}n^6 & -\tfrac{21}{64}n^7 & +\tfrac{28}{128}n^8 & -\tfrac{36}{256}n^9 & \\
+\tfrac{10}{32}n^6 & -\tfrac{35}{64}n^7 & +\tfrac{56}{128}n^8 & -\tfrac{84}{256}n^9 & & & \\
+\tfrac{35}{128}n^8 & & & & & &
\end{array}$$

vnde patet, ſi ponatur

$$\frac{1}{1 + n\,\text{coſ.}\,\Phi} = A - B\,\text{coſ.}\,\Phi + C\,\text{coſ.}\,2\Phi - D\,\text{coſ.}\,3\Phi$$
$$+ E\,\text{coſ.}\,4\Phi - \text{etc. eſt}$$

A =

$$A = 1 + \tfrac{1}{2} nn + \tfrac{3}{8} n^4 + \tfrac{10}{32} n^6 + \text{etc. seu}$$

$$A = 1 + \tfrac{1}{2} n^2 + \tfrac{1.3}{2.4} n^4 + \tfrac{1.3.5}{2.4.6} n^6 + \tfrac{1.3.5.7}{2.4.6.8} n^8 + \text{etc.}$$

ſicque euidens eſt eſſe $A = \frac{1}{\sqrt{(1 - nn)}}$.

Simili modo eſt

$$B = n + \tfrac{3}{4} n^3 + \tfrac{10}{16} n^5 + \text{etc.} = \tfrac{2}{n} \left(\tfrac{1}{2} n^2 + \tfrac{1.3}{2.4} n^4 + \tfrac{1.3.5}{2.4.6} n^5 + \text{etc.}\right)$$

ideoque $B = \frac{2}{n} \left(\frac{1}{\sqrt{(1 - nn)}} - 1\right)$. Verum et hunc valorem et ſequentes facilius hoc modo definire licet. Cum ſit

$$\frac{1}{1 + n \operatorname{coſ.} \Phi} = A - B \operatorname{coſ.} \Phi + C \operatorname{coſ.} 2\Phi - D \operatorname{coſ.} 3\Phi$$
$$+ E \operatorname{coſ.} 4\Phi - \text{etc.}$$

multiplicetur per $1 + n \operatorname{coſ.} \Phi$, et quia

$\operatorname{coſ.} \Phi \operatorname{coſ.} \lambda\Phi = \frac{1}{2} \operatorname{coſ.} (\lambda - 1)\Phi + \frac{1}{2} \operatorname{coſ.} (\lambda + 1)\Phi$, fiet

$$\begin{array}{lllll}
1 = A & - B \operatorname{coſ.}\Phi & + C \operatorname{coſ.} 2\Phi & - D \operatorname{coſ.} 3\Phi & + E \operatorname{coſ.} 4\Phi - \text{etc.} \\
 & + A n & - \tfrac{1}{2} B n & + \tfrac{1}{2} C n & - \tfrac{1}{2} D n \\
 - \tfrac{1}{2} B n + \tfrac{1}{2} C n & - \tfrac{1}{2} D n & + \tfrac{1}{2} E n & - \tfrac{1}{2} F n &
\end{array}$$

vnde quia A iam definiuimus, reliqui coefficientes ita determinantur:

$$\begin{array}{ll}
B = \frac{2}{n}(A - 1); & E = \frac{2D - Cn}{n} \\
C = \frac{2B - An}{n}; & F = \frac{2E - Dn}{n} \\
D = \frac{2C - Bn}{n}; & G = \frac{2F - En}{n}
\end{array}$$

etc.

His igitur coefficientibus inuentis, integrale facile aſſignatur: nam cum ſit $\int \partial\Phi \operatorname{coſ.} \lambda\Phi = \frac{1}{\lambda} \operatorname{ſin.} \lambda\Phi$, habebimus

$$\int \frac{\partial\Phi}{1 + n \operatorname{coſ.} \Phi} = A\Phi - B \operatorname{ſin.} \Phi + \tfrac{1}{2} C \operatorname{ſin.} 2\Phi$$
$$- \tfrac{1}{3} D \operatorname{ſin.} 3\Phi + \tfrac{1}{4} E \operatorname{ſin.} 4\Phi - \text{etc.}$$

quae ſeries ſecundum ſinus angulorum Φ, 2Φ, 3Φ, etc. progreditur, vti deſiderabatur.

Corollarium 1.

273. Primo patet hanc resolutionem locum habere non posse, nisi n sit numerus vnitate minor; si enim $n > 1$, singuli coefficientes prodeunt imaginarii. Sin autem sit $n = 1$, ob $1 + \operatorname{cos}. \Phi = 2 \operatorname{cos}. \frac{1}{2}\Phi^2$, erit integrale

$$\int \frac{\partial \Phi}{1 + \operatorname{cos}. \Phi} = \int \frac{\frac{1}{2}\partial \Phi}{\operatorname{cos}. \frac{1}{2}\Phi^2} = \operatorname{tang}. \tfrac{1}{2}\Phi.$$

Corollarium 2.

274. Cum sit $A = \frac{1}{\sqrt{(1-nn)}}$ et $B = \frac{2}{n}\left(\frac{1}{\sqrt{(1-nn)}} - 1\right)$, reliqui coefficientes C, D, E, etc. seriem recurrentem constituunt, ita vt si bini contigui sint P et Q sequens futurus sit $\frac{2}{n}Q - P$. Hinc cum aequationis $zz = \frac{2}{n}z - 1$ radices sint $\frac{1 \pm \sqrt{(1-nn)}}{n}$, quisque terminus in hac forma continetur

$$\alpha\left(\frac{1+\sqrt{(1-nn)}}{n}\right)^\lambda + \beta\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^\lambda.$$

Corollarium 3.

275. Quia autem in nostra lege non A sed $2A$ sumitur, posito $\lambda = 0$, prodire debet $2A$, ideoque $\alpha + \beta = \frac{2}{\sqrt{(1-nn)}}$: deinde facto $\lambda = 1$, fieri debet

$$\frac{\alpha+\beta}{n} + \frac{(\alpha-\beta)\sqrt{(1-nn)}}{n} = \frac{2 - 2(1-nn)}{n\sqrt{(1-nn)}},$$

vnde $\alpha - \beta = -\frac{2}{\sqrt{(1-nn)}}$. Ergo $\alpha = 0$ et $\beta = \frac{2}{\sqrt{(1-nn)}}$; sicque quilibet terminus praeter A erit $=$

$$\frac{2}{\sqrt{(1-nn)}}\left(\frac{1-\sqrt{(1-nn)}}{n\sqrt{(1-nn)}}\right)^\lambda.$$

Corollarium 4.

276. Coefficientes ergo euoluti ita se habebunt:

$A =$

$$A = \frac{1}{\sqrt{(1 - nn)}}$$
$$B = \frac{2 - 2\sqrt{(1 - nn)}}{n\sqrt{(1 - nn)}}$$
$$C = \frac{4 - 2nn - 4\sqrt{(1 - nn)}}{nn\sqrt{(1 - nn)}}$$
$$D = \frac{8 - 6nn - 2(4 - nn)\sqrt{(1 - nn)}}{n^3\sqrt{(1 - nn)}}$$
$$E = \frac{16 - 16nn + 2n^4 - 2(8 - 4nn)\sqrt{(1 - nn)}}{n^4\sqrt{(1 - nn)}}$$
$$F = \frac{32 - 40nn + 10n^4 - 2(16 - 12nn + n^4)\sqrt{(1 - nn)}}{n^5\sqrt{(1 - nn)}}$$
$$G = \frac{64 - 96nn + 36n^4 - 2n^6 - 2(32 - 23nn + 6n^4)\sqrt{(1 - nn)}}{n^6\sqrt{(1 - nn)}}$$

Corollarium 5.

277. Quia $n < 1$, hi coefficientes plerumque facilius determinantur per ſeries primum inuentas, ſcilicet:

$$A = 1 + \tfrac{1}{2}n^2 + \tfrac{1.3}{2.4}n^4 + \tfrac{1.3.5}{2.4.6}n^6 + \tfrac{1.3.5.7}{2.4.6.8}n^8 + \text{etc.}$$
$$B = n\,(1 + \tfrac{3}{4}n^2 + \tfrac{3.5}{4.6}n^4 + \tfrac{3.5.7}{4.6.8}n^6 + \tfrac{3.5.7.9}{4.6.8.10}n^8 + \text{etc.}$$
$$C = \tfrac{1}{2}n^2\,(1 + \tfrac{3.4}{2.6}n^2 + \tfrac{3.4.5.6}{2.6.4.8}n^4 + \tfrac{3.4.5.6.7.8}{2.6.4.8.6.10}n^6 + \text{etc.}$$
$$D = \tfrac{1}{4}n^3\,(1 + \tfrac{4.5}{2.8}n^2 + \tfrac{4.5.6.7}{2.8.4.10}n^4 + \tfrac{4.5.6.7.8.9}{2.8.4.10.6.12}n^6 + \text{etc.}$$
$$E = \tfrac{1}{8}n^4\,(1 + \tfrac{5.6}{2.10}n^2 + \tfrac{5.6.7.8}{2.10.4.12}n^4 + \tfrac{5.6.7.8.9.10}{2.10.4.12.6.14}n^6 + \text{etc.}$$
$$F = \tfrac{1}{16}n^6\,(1 + \tfrac{6.7}{2.12}n^2 + \tfrac{6.7.8.9}{2.12.4.14}n^4 + \tfrac{6.7.8.9.10.11}{2.12.4.14.6.16}n^6 + \text{etc.}$$

etc.

Scholion.

278. Cum ex his valoribus fit

$$\int \frac{\partial\Phi}{1 + n\cos.\Phi} = A\Phi - B\,\text{fin.}\,\Phi + \tfrac{1}{2}C\,\text{fin.}\,2\Phi$$
$$- \tfrac{1}{3}D\,\text{fin.}\,3\Phi + \tfrac{1}{4}E\,\text{fin.}\,4\Phi - \text{etc.}$$

in hac ſerie terminus primus $A\Phi$ imprimis eſt notandus, quod creſcente angulo Φ continuo creſcat, idque in infinitum vsque, dum reliqui termini modo creſcent modo decreſcent: neque

que tamen certum limitem excedunt; nam ſin. $\lambda\Phi$ neque ſupra $+1$ creſcere, neque infra -1 decreſcere poteſt. Cum deinde hoc integrale ſupra inuentum ſit

$$\frac{1}{\sqrt{(1-nn)}} \text{ Ang. cof. } \frac{n+\cos.\Phi}{1+n\cos.\Phi}$$

ſeries illa huic angulo aequatur. Quare ſi hic angulus vocetur ω, vt ſit $\partial\omega = \frac{\partial\Phi\sqrt{(1-nn)}}{1+n\cos.\Phi}$, erit $\cos.\omega = \frac{n+\cos.\Phi}{1+n\cos.\Phi}$, hincque $n+\cos.\Phi-\cos.\omega-n\cos.\Phi\cos.\omega=0$, ex quo eſt viciſſim $\cos.\Phi=\frac{\cos.\omega-n}{1-n\cos.\omega}$, quae formula cum ex illa naſcatur ſumto n negatiuo, erit

$\partial\Phi=\frac{\partial\omega\sqrt{(1-nn)}}{1-n\cos.\Phi}$, et

$$\frac{\Phi}{\sqrt{(1-nn)}}=A\omega+B\sin.\omega+\tfrac{1}{2}C\sin.2\omega+\tfrac{1}{3}D\sin.3\omega+\tfrac{1}{4}E\sin.4\omega+\text{etc.}$$

Quia vero eſt

$$\frac{\omega}{\sqrt{(1-nn)}}=A\Phi-B\sin.\Phi+\tfrac{1}{2}C\sin.2\Phi-\tfrac{1}{3}D\sin.3\Phi+\tfrac{1}{4}E\sin.4\Phi-\text{etc.}$$

ob $\frac{1}{\sqrt{(1-nn)}}=A$, habebimus:

$$0=B(\sin.\omega-\sin.\Phi)+\tfrac{1}{2}C(\sin.2\omega+\sin.2\Phi)+\tfrac{1}{3}D(\sin.3\omega-\sin.3\Phi)+\text{etc.}$$

cuiusmodi relationes notaſſe iuuabit.

Problema 36.

279. Integrale formulae $\partial\Phi(1+n\cos.\Phi)^\nu$ per ſeriem, ſecundum ſinus angulorum multiplorum ipſius Φ progredientem, exprimere.

Solutio.

Cum ſit

$$(1+n\cos.\Phi)^\nu = 1+\tfrac{\nu}{1}n\cos.\Phi+\tfrac{\nu(\nu-1)}{1.2}n^2\cos.\Phi^2+\tfrac{\nu(\nu-1)(\nu-2)}{1.2.3}n^3\cos.\Phi^3+\text{etc.}$$

ſi

ſi ponamus

$$(1 + n \operatorname{cof.} \Phi)^\nu = A + B \operatorname{cof.} \Phi + C \operatorname{cof.} 2\Phi + D \operatorname{cof.} 3\Phi + E \operatorname{cof.} 4\Phi + \text{etc.}$$

erit per formulas ſupra indicatas:

$$A = 1 + \frac{\nu(\nu-1)}{1 \cdot 2} \cdot \frac{1}{2} n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{1 \cdot 3}{2 \cdot 4} n^4 + \frac{\nu(\nu-1) \ldots\ldots (\nu-5)}{1 \cdot 2 \ldots\ldots 6} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} n^6 + \text{etc.}$$

$$B = 2n \left[\frac{\nu}{2} + \frac{\nu(\nu-1)(\nu-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1 \cdot 3}{2 \cdot 4} n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} n^4 + \text{etc.}\right.$$

quae ſeries ita clarius exhibentur:

$$A = 1 + \frac{\nu(\nu-1)}{2 \cdot 2} n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2 \cdot 2 \cdot 4 \cdot 4} n^4 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} n^6 + \text{etc.}$$

$$\frac{1}{2} B = \frac{\nu}{2} n + \frac{\nu(\nu-1)(\nu-2)}{2 \cdot 2 \cdot 4} n^3 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} n^5 + \text{etc.}$$

Inuentis autem his binis coefficientibus A et B, reliqui ex his ſequenti modo commodius determinari poterunt. Cum ſit

$$\nu\, l(1 + n \operatorname{cof.} \Phi) = l[A + B \operatorname{cof.} \Phi + C \operatorname{cof.} 2\Phi + D \operatorname{cof.} 3\Phi + E \operatorname{cof.} 4\Phi + \text{etc.}]$$

ſumantur differentialia, ac per $-\partial\Phi$ diuidendo prodit

$$\frac{\nu n \operatorname{fin.} \Phi}{1 + n \operatorname{cof.} \Phi} = \frac{B \operatorname{fin.} \Phi + 2C \operatorname{fin.} 2\Phi + 3D \operatorname{fin.} 3\Phi + 4E \operatorname{fin.} 4\Phi + \text{etc.}}{A + B \operatorname{cof.} \Phi + C \operatorname{cof.} 2\Phi + D \operatorname{cof.} 3\Phi + E \operatorname{cof.} 4\Phi + \text{etc.}}.$$

Iam per crucem multiplicando,

ob $\operatorname{fin.} \lambda\Phi \operatorname{cof.} \Phi = \frac{1}{2} \operatorname{fin.} (\lambda+1)\Phi + \frac{1}{2} \operatorname{fin.} (\lambda-1)\Phi$ et

$\operatorname{fin.} \Phi \operatorname{cof.} \lambda\Phi = \frac{1}{2} \operatorname{fin.} (\lambda+1)\Phi - \frac{1}{2} \operatorname{fin.} (\lambda-1)\Phi$,

peruenietur ad hanc aequationem:

$$\begin{array}{llllll}
0 = B \operatorname{fin.} \Phi & + 2C \operatorname{fin.} 2\Phi & + 3D \operatorname{fin.} 3\Phi & + 4E \operatorname{fin.} 4\Phi & + 5F \operatorname{fin.} 5\Phi & + \text{etc.} \\
 & + \frac{1}{2} Bn & + \frac{2}{2} Cn & + \frac{3}{2} Dn & + \frac{4}{2} En & \\
+ \frac{2}{2} Cn & + \frac{3}{2} Dn & + \frac{4}{2} En & + \frac{5}{2} Fn & + \frac{6}{2} Gn & \\
- \nu An & - \frac{\nu}{2} Bn & - \frac{\nu}{2} Cn & - \frac{\nu}{2} Dn & - \frac{\nu}{2} En & \\
+ \frac{\nu}{2} Cn & + \frac{\nu}{2} Dn & + \frac{\nu}{2} En & + \frac{\nu}{2} Fn & + \frac{\nu}{2} Gn &
\end{array}$$

 vnde

vnde hae ſequuntur determinationes :

$$\begin{array}{l|l}
(\nu+2)\,\mathrm{C}n + 2\,\mathrm{B} - 2\nu\mathrm{A}n = 0 & \mathrm{C} = \frac{2\nu\mathrm{A}n - 2\mathrm{B}}{(\nu+2)n} \\
(\nu+3)\,\mathrm{D}n + 4\,\mathrm{C} - (\nu-1)\,\mathrm{B}n = 0 & \mathrm{D} = \frac{(\nu-1)\mathrm{B}n - 4\mathrm{C}}{(\nu+3)n} \\
(\nu+4)\,\mathrm{E}n + 6\,\mathrm{D} - (\nu-2)\,\mathrm{C}n = 0 & \mathrm{E} = \frac{(\nu-2)\mathrm{C}n - 6\mathrm{D}}{(\nu+4)n} \\
(\nu+5)\,\mathrm{F}n + 8\,\mathrm{E} - (\nu-3)\,\mathrm{D}n = 0 & \mathrm{F} = \frac{(\nu-3)\mathrm{D}n - 8\mathrm{E}}{(\nu+5)n} \\
(\nu+6)\,\mathrm{G}n + 10\,\mathrm{F} - (\nu-4)\,\mathrm{E}n = 0 & \mathrm{G} = \frac{(\nu-4)\mathrm{E}n - 10\mathrm{F}}{(\nu+6)n}
\end{array}$$

vbi ſi ſuperiores valores pro A et B ſubſtituantur, reperitur:

$$\mathrm{C} = 4nn\left(\frac{1\nu(\nu-1)}{2.\,2.\,4} + \frac{2\nu(\nu-1)(\nu-2)(\nu-3)}{2.\,2.\,4.\,4.\,6}n^2 + \frac{3\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2.\,2.\,4.\,4.\,6.\,6.\,8}n^4 - \text{etc.}\right.$$

$$\mathrm{D} = 8n^3\left(\frac{1.\,2\nu(\nu-1)(\nu-2)}{2.\,2.\,4.\,4.\,6} + \frac{2.\,3\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2.\,2.\,4.\,4.\,6.\,6.\,8}n^2 + \text{etc.}\right.$$

$$\mathrm{E} = 16n^4\left(\frac{1.\,2.\,3\nu(\nu-1)(\nu-2)(\nu-3)}{2.\,2.\,4.\,4.\,6.\,6.\,8} + \frac{2.\,3.\,4\nu(\nu-1)\ldots\ldots(\nu-5)}{2.\,2.\,4.\,4.\,6.\,6.\,8.\,8.\,10}n^2 + \text{etc.}\right.$$

etc.

vnde forma ſequentium ſerierum colligitur.

His autem inuentis coefficientibus, erit integrale quaeſitum

$$\int \partial\Phi(1 + n\,\mathrm{cos}.\,\Phi)^\nu = \mathrm{A}\Phi + \mathrm{B}\,\mathrm{ſin}.\,\Phi + \tfrac{1}{2}\mathrm{C}\,\mathrm{ſin}.\,2\Phi + \tfrac{1}{3}\mathrm{D}\,\mathrm{ſin}.\,3\Phi + \tfrac{1}{4}\mathrm{E}\,\mathrm{ſin}.\,4\Phi + \text{etc.}$$

Corollarium 1.

280. Ad ſimilitudinem harum ſerierum pro C, D, E etc. datarum, etiam valor ipſius B ita exprimi poteſt:

$$\mathrm{B} = 2n\left(\frac{\nu}{2} + \frac{\nu(\nu-1)(\nu-2)}{2.\,2.\,4}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2.\,2.\,4.\,4.\,6}n^4 + \text{etc.}\right.$$

Series autem pro A inuenta formam habet ſingularem in hac lege non comprehenſam.

Corollarium 2.

281. Si ſeries A et B inter ſe comparemus, varias relationes inter eas obſeruare licet, quarum haec primo ſe offert:

A n

$$A\,n + \tfrac{1}{2}B = \frac{(\nu+2)}{2}\,n\left[1 + \frac{\nu(\nu-1)}{2\cdot 4}\,n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 4\cdot 4\cdot 6}\,n^4 + \frac{\nu(\nu-1)\ldots\ldots(\nu-5)}{2\cdot 4\ldots\ldots 8}\,n^6 + \text{etc.}\right]$$

quae a ferie A tantum fecundum denominatores differ.

Corollarium 3.

282. Ponamus $\frac{2Ann + Bn}{\nu+2} = N$, vt fit

$$N = n^2 + \frac{\nu(\nu-1)}{2\cdot 4}\,n^4 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 4\cdot 4\cdot 6}\,n^6 \text{ etc.}$$

$$A = 1 + \frac{\nu(\nu-1)}{2\cdot 2}\,n^2 + \frac{\nu(\nu-1)(\nu-1)(\nu-3)}{2\cdot 2\cdot 4\cdot 4}\,n^4 \text{ etc.}$$

Quodfi iam n vt variabilis tractetur, differentiatio praebet:

$$\frac{\partial N}{n\,\partial n} = 2 + \frac{\nu(\nu-1)}{2}\,n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 4\cdot 4}\,n^4 + \text{etc.} = 2A.$$

Cum igitur fit

$$\partial N = \frac{4An\,\partial n + B\,\partial n + 2nn\,\partial A + n\,\partial B}{\nu+2} = 2An\,\partial n,$$

erit $2\nu An\,\partial n = 2nn\,\partial A + B\,\partial n + n\,\partial B$.

Corollarium 4.

283. Ex dato ergo coefficiente A, coefficiens B ita per integrationem inueniri poteft, vt fit

$$Bn = 2\int(\nu An\,\partial n - nn\,\partial A):$$

vel erit etiam ex illa forma

$$B = \frac{2(\nu+2)}{n}\int An\,\partial n - 2An.$$

Vbi notandum eft, pofito $n = 0$ integrale $\int An\,\partial n$ euanefcere debere, quia hoc cafu B euanefcit.

Scholion.

284. Series pro litteris B, C, D, etc. inuentas etiam fequenti modo per continuos factores exprimere licet:

$$B = \nu n\left(1 + \frac{(\nu-1)}{2\cdot}\frac{(\nu-2)}{4}\,n^2 + \frac{(\nu-3)(\nu-4)}{4\cdot\;\;6}\,Pn^2 + \frac{(\nu-5)}{6\cdot}\frac{(\nu-6)}{8}\,Pn^2 + \text{etc.}\right.$$

$$C = \frac{\nu(\nu-1)}{1.\ 2} \cdot \frac{n^2}{2}\Big(1 + \frac{(\nu-2)(\nu-3)}{2.\ 6}n^2 + \frac{(\nu-4)(\nu-5)}{4.\ 8}Pn^2$$
$$+ \frac{(\nu-6)(\nu-7)}{6.\ 10}Pn^2 + \text{etc.}$$

$$D = \frac{\nu(\nu-1)(\nu-2)}{1.\ 2.\ 3} \cdot \frac{n^3}{4}\Big(1 + \frac{(\nu-3)(\nu-4)}{2.\ 8}n^2 + \frac{(\nu-5)(\nu-6)}{4.\ 10}Pn^2$$
$$+ \frac{(\nu-7)(\nu-8)}{6.\ 12}Pn^2 + \text{etc.}$$

$$E = \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1.\ 2.\ 3.\ 4} \cdot \frac{n^4}{8}\Big(1 + \frac{(\nu-4)(\nu-5)}{2.\ 10}n^2 + \frac{(\nu-6)(\nu-7)}{4.\ 12}Pn^2$$
$$+ \frac{(\nu-8)(\nu-9)}{6.\ 14}Pn^2 + \text{etc.}$$

$$F = \frac{\nu \ldots\ldots (\nu-4)}{1 \ldots\ldots 5} \cdot \frac{n^5}{18}\Big(1 + \frac{(\nu-5)(\nu-6)}{1\ .\ 12}n^2 + \frac{(\nu-7)(\nu-8)}{4\ .\ 14}Pn^2$$
$$+ \frac{(\nu-9)(\nu-10)}{6.\ 16}Pn^2 + \text{etc.}$$

etc.

vbi in qualibet ferie littera P terminum praecedentem integrum denotat. Atque ope ferierum iftarum coefficientes plerumque facilius inueniuntur, quam ex lege ante tradita, qua quisque ex binis praecedentibus determinatur. Quin haec lex defectu laborat, quod fi ν fuerit numerus integer negatiuus praeter -1, quidam coefficientes plane non definiantur, quos ergo ex his feriebus defumi oportet. Ita fi fuerit

$$\nu = -2, \text{ erit } B = \nu A n = -2An, \text{ et}$$

$$C = \frac{3}{1} \cdot \frac{n^2}{2}\Big(1 + \frac{4.\ 5}{2.\ 6}n^2 + \frac{4.\ 5.\ 6.\ 7}{2.\ 6.\ 4.\ 8}n^4 + \frac{4.\ 5.\ 6.\ 7.\ 8.\ 9}{2.\ 6.\ 4.\ 8.\ 6.\ 10}n^6 + \text{etc.}\Big)$$

fi fit $\nu = -3$, erit $C = -Bn$, et

$$D = -\frac{4.\ 5}{1.\ 2} \cdot \frac{n^3}{4}\Big(1 + \frac{6.\ 7}{2.\ 3}n^2 + \frac{6.\ 7.\ 8.\ 9}{2.\ 8.\ 4.\ 10}n^4 + \frac{6.\ 7.\ 8.\ 9.\ 10.\ 11}{2.\ 8.\ 4.\ 10.\ 6.\ 12}n^6 + \text{etc.}\Big)$$

fi fit $\nu = -4$, erit $D = -Cn$, et

$$E = \frac{5.\ 6.\ 7}{1.\ 2.\ 3} \cdot \frac{n^4}{8}\Big(1 + \frac{8.\ 9}{2.\ 10}n^2 + \frac{8.\ 9.\ 10.\ 11}{2.\ 10.\ 4.\ 12}n^4 + \frac{8.\ 9.\ 10.\ 11.\ 12.\ 13}{2.\ 10.\ 4.\ 12.\ 6.\ 14}n^6 + \text{etc.}\Big)$$

fi fit $\nu = -5$, erit $E = -Dn$, et

$$F = -\frac{6.\ 7.\ 8.\ 9}{1.\ 2.\ 3.\ 4}\Big(1 + \frac{10.\ 11}{2.\ 12}n^2 + \frac{10.\ 11.\ 12.\ 13}{2.\ 12.\ 4.\ 14}n^4$$
$$+ \frac{10.\ 11.\ 12.\ 13.\ 14.\ 15}{2.\ 12.\ 4.\ 14.\ 6.\ 16}n^6 + \text{etc.}$$

et ita de reliquis.

Exem-

Exemplum 1.

285. *Formulae* $\partial\Phi\,(1+n\cos.\Phi)^\nu$ *integrale euoluere, si* ν *sit numerus integer positiuus.*

$$\text{Posito } (1+n\cos.\Phi)^\nu = A + B\cos.\Phi + C\cos.2\Phi + D\cos.3\Phi + E\cos.4\Phi + \text{etc.}$$

pro singulis valoribus exponentis ν habebimus:

1.) si $\nu = 1$; $A = 1$; $B = 2$; $C = 0$; etc.

2.) si $\nu = 2$; $A = 1 + \frac{1}{2}n^2$; $B = 2n$; $C = \frac{1}{2}nn$; $D = 0$; etc.

3.) si $\nu = 3$; $A = 1 + \frac{3}{2}n^2$; $B = 3n(1 + \frac{1}{4}n^2)$; $C = \frac{3}{2}n^2$; $D = \frac{1}{4}n^3$; $E = 0$; etc.

4.) si $\nu = 4$; $A = 1 + \frac{6}{2}n^2 + \frac{3}{8}n^4$; $B = 4n(1 + \frac{3}{4}n^2)$; $C = 3n^2(1 + \frac{1}{6}n^2)$; $D = n^3$; $E = \frac{1}{8}n^4$; $F = 0$.

Hi autem casus nihil habent difficultatis. Ad vsum sequentem tantum iuuabit primum terminum absolutum A notasse:

si $\nu = 1$; $A = 1$;

si $\nu = 2$; $A = 1 + \frac{2.1}{2.2}n^2$;

si $\nu = 3$; $A = 1 + \frac{3.2}{2.2}n^2$;

si $\nu = 4$; $A = 1 + \frac{4.3}{2.2}n^2 + \frac{4.3.2.1}{2.2.4.4}n^4$;

si $\nu = 5$; $A = 1 + \frac{5.4}{2.2}n^2 + \frac{5.4.3.2}{2.2.4.4}n^4$;

si $\nu = 6$; $A = 1 + \frac{6.5}{2.2}n^2 + \frac{6.5.4.3}{2.2.4.4}n^4 + \frac{6.5.4.3.2.1}{2.2.4.4.6.6}n^6$;

si $\nu = 7$; $A = 1 + \frac{7.6}{2.2}n^2 + \frac{7.6.5.4}{2.2.4.4}n^4 + \frac{7.6.5.4.3.2}{2.2.4.4.6.6}n^6$;

etc.

Exemplum 2.

286. *Formulae* $\dfrac{\partial\Phi}{(1+n\cos.\Phi)^\mu}$ *integrale per seriem euoluere.*

Posito $\frac{1}{(1+n \operatorname{cof.} \Phi)^{\mu}} = A + B \operatorname{cof.} \Phi + C \operatorname{cof.} 2\Phi$
$+ D \operatorname{cof.} 3\Phi + E \operatorname{cof.} 4\Phi + \text{etc.}$

ex praecedentibus formulis ponendo $\nu = -\mu$ erit

$$A = 1 + \frac{\mu(\mu+1)}{2 \cdot 2} n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2 \cdot 2 \cdot 4 \cdot 4} n^4$$
$$+ \frac{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} n^6 + \text{etc.}$$

$$B = -\mu n \left(1 + \frac{(\mu+1)(\mu+2)}{2 \cdot 4} n^2 + \frac{(\mu+3)(\mu+4)}{4 \cdot 6} P n^2\right.$$
$$\left. + \frac{(\mu+5)(\mu+6)}{6 \cdot 8} P n^2 + \text{etc.}\right);$$

$$C = \frac{\mu(\mu+1)}{1 \cdot 2} \cdot \frac{n^2}{2} \left(1 + \frac{(\mu+2)(\mu+3)}{2 \cdot 6} n^2 + \frac{(\mu+4)(\mu+5)}{4 \cdot 8} P n^2\right.$$
$$\left. + \frac{(\mu+6)(\mu+7)}{6 \cdot 10} P n^2 + \text{etc.}\right);$$

$$D = -\frac{\mu(\mu+1)(\mu+2)}{1 \cdot 2 \cdot 3} \cdot \frac{n^3}{4} \left(1 + \frac{(\mu+3)(\mu+4)}{2 \cdot 8} n^2 + \frac{(\mu+5)(\mu+6)}{4 \cdot 10} P n^2\right.$$
$$\left. + \frac{(\mu+7)(\mu+8)}{6 \cdot 12} P n^2 + \text{etc.}\right);$$

etc.

vbi vt ante in quaque ferie P terminum praecedentem denotat. Hi autem coefficientes ita a fe inuicem pendent, vt fit

$$B = \frac{-2(\mu-2)}{n} \int A n \partial n - 2 A n \text{ et}$$

$$C = \frac{2B + 2\mu A n}{(\mu-2) n}; \qquad D = \frac{4C + (\mu+1) B n}{(\mu-3) n};$$
$$E = \frac{6D + (\mu+2) C n}{(\mu-4) n}; \qquad F = \frac{8E + (\mu+3) D n}{(\mu-5) n};$$
$$G = \frac{10F + (\mu+4) E n}{(\mu-6) n}; \qquad H = \frac{12G + (\mu+5) F n}{(\mu-7) n};$$

etc.

Vbi incommodo, quando μ eft numerus integer, fupra iam remedium eft allatum. Hic igitur praecipue inueftigamus quomodo coefficientes cuiusque cafus ex cafu praecedente determinari queant, quod ita fieri poterit. Cum fit

$$\frac{1}{(1+n \operatorname{cof.} \Phi)^{\mu}} = A + B \operatorname{cof.} \Phi + C \operatorname{cof.} 2\Phi$$
$$+ D \operatorname{cof.} 3\Phi + \text{etc.}$$

po-

ponatur

$$\frac{1}{(1+n\,\text{cos}.\,\phi)^{\mu+1}} = A' + B'\,\text{cos}.\,\phi + C'\,\text{cos}.\,2\,\phi + D'\,\text{cos}.\,3\,\phi + \text{etc.}$$

haec igitur series per $1 + n\,\text{cos}.\,\phi$ multiplicata in illam abire debet, est autem productum

$$\begin{array}{llll} A' & + B'\,\text{cos}.\,\phi + & C'\,\text{cos}.\,2\,\phi + & D'\,\text{cos}.\,3\,\phi + \text{etc.} \\ & + A'n & + \tfrac{1}{2}B'n & + \tfrac{1}{2}C'n \\ + \tfrac{1}{2}B'n + \tfrac{1}{2}C'n & & + \tfrac{1}{2}D'n & + \tfrac{1}{2}E'n \end{array}$$

vnde colligimus

$$B' = \frac{2(A-A')}{n}; \qquad C' = \frac{2(B-B')-2A'n}{n};$$

$$D' = \frac{2(C-C')-B'n}{n}; \quad E' = \frac{2(D-D')-C'n}{n}; \text{ etc.}$$

dummodo ergo coefficiens A' constaret, sequentes B', C', D' etc. haberemus. Videamus igitur quomodo A' ex A determinari possit: quia est

$$A = 1 + \frac{\mu(\mu+1)}{2\cdot 2}n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.}$$

$$A' = 1 + \frac{(\mu+1)(\mu+2)}{2\cdot 2}n^2 + \frac{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.}$$

tractetur n vt variabilis, ac prior series per n^μ multiplicata differentietur, vt prodeat

$$\frac{\partial A n^\mu}{\partial n} = \mu n^{\mu-1} + \frac{\mu(\mu+1)(\mu+2)n^{\mu+1}}{2\cdot 2} + \frac{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2\cdot 2\cdot 4\cdot 4}n^{\mu+3} + \text{etc.}$$

quae series manifesto est $= \mu n^{\mu-1} A'$; quocirca A' ita per A determinatur, vt sit

$$A' = \frac{\partial\,.\,A n^\mu}{\partial\,.\,n^\mu} = A + \frac{n\,\partial A}{\mu\,\partial n}.$$

Cum igitur pro casu $\mu = 1$ inuenerimus

$A =$

$$A = \frac{1}{\sqrt{(1 - nn)}}; \text{ ob } \frac{\partial A}{\partial n} = \frac{n}{(1 - nn)^{\frac{3}{2}}}; \text{ erit}$$

$$A' = \frac{1}{\sqrt{(1 - nn)}} + \frac{nn}{(1 - nn)^{\frac{3}{2}}} = \frac{1}{(1 - nn)^{\frac{3}{2}}}.$$

Hic iam eſt valor ipſius A pro $\mu = 2$, vnde ob

$$\frac{\partial A}{\partial n} = \frac{3n}{(1 - nn)^{\frac{5}{2}}}, \text{ fiet pro } \mu = 3,$$

$$A = \frac{1}{(1 - an)^{\frac{3}{2}}} + \frac{3nn}{2(1 - nn)^{\frac{5}{2}}} = \frac{1 + \frac{1}{2}nn}{(1 - nn)^{\frac{5}{2}}}.$$

Hoc modo ſi vlterius progrediamur, reperiemus:

$$\text{ſi } \mu = 1; \; A = \frac{1}{\sqrt{(1 - nn)}};$$

$$\text{ſi } \mu = 2; \; A = \frac{1}{(1 - nn)\sqrt{(1 - nn)}};$$

$$\text{ſi } \mu = 3; \; A = \frac{1 + \frac{1}{2}nn}{(1 - nn)^2\sqrt{(1 - nn)}};$$

$$\text{ſi } \mu = 4; \; A = \frac{1 + \frac{3}{2}nn}{(1 - nn)^3\sqrt{(1 - nn)}};$$

$$\text{ſi } \mu = 5; \; A = \frac{1 + 3nn + \frac{3}{8}n^4}{(1 - nn)^4\sqrt{(1 - nn)}}.$$

Corollarium 1.

287. Eodem modo etiam reliqui coefficientes B', C' etc. ex analogis B, C etc. definientur, eruntque omnes iſtae relationes inter ſe ſimiles, ſcilicet vti eſt

$A =$

$$A' = \frac{\partial.\, A\, n^{\mu}}{\partial.\, n^{\mu}} = A + \frac{n\,\partial A}{\mu\,\partial n}, \text{ ita erit}$$

$$B' = \frac{\partial\, B\, n^{\mu}}{\partial.\, n^{\mu}} = B + \frac{n\,\partial B}{\mu\,\partial n};$$

$$C' = \frac{\partial.\, C\, n^{\mu}}{\partial.\, n^{\mu}} = C + \frac{n\,\partial C}{\mu\,\partial n};$$

etc.

Corollarium 2.

288. At ante inuenimus $B' = \frac{2(A - A')}{n}$, vnde fiet

$B' = -\frac{2\,\partial A}{\mu\,\partial n} = B + \frac{n\,\partial B}{\mu\,\partial n}$, hincque

$$\mu\, B\,\partial n + n\,\partial B + 2\,\partial A = 0:$$

mult. per $n^{\mu - 1}$ vt fit

$$\partial.\, B\, n^{\mu} + 2\, n^{\mu - 1}\,\partial A = 0,$$

vnde integrando

$$B\, n^{\mu} = -2 \int n^{\mu - 1}\,\partial A = -2\, n^{\mu - 1}\, A + 2\,(\mu - 1)\int A\, n^{\mu - 2}\,\partial n:$$

ideoque

$$B = -\frac{2\, A}{n} + \frac{2\,(\mu - 1)}{n^{\mu}}\int A\, n^{\mu - 2}\,\partial n.$$

At ante habueramus

$$B = -2\, A\, n - \frac{2\,(\mu - 2)}{n}\int A\, n\,\partial n.$$

Corollarium 3.

289. His valoribus aequatis, obtinetur aequatio inter A et n, qua quantitas A per n determinatur, erit enim

$$n^{-\mu}\int n^{\mu - 1}\,\partial A = A\, n + \frac{(\mu - 2)}{n}\int A\, n\,\partial n:$$

vnde per duplicem differentiationem prodit

$$(1 - nn)\,\partial\partial A + \frac{\partial n\,\partial A}{n} - 2\,(\mu + 1)\, n\,\partial n\,\partial A - \mu\,(\mu + 1)\, A\,\partial n^2 = 0.$$

Scholion 1.

290. Si hos valores ipsius A cum superioribus, vbi μ erat numerus integer negatiuus, inter se comparemus, eximiam conuenientiam deprehendemus.

Pro superioribus.	Pro his formulis.
si $\nu=0$; $A=1$	si $\mu=1$; $A=\frac{1}{\sqrt{(1-nn)}}$
$\nu=1$; $A=1$	$\mu=2$; $A=\frac{1}{(1-nn)\sqrt{(1-nn)}}$
$\nu=2$; $A=1+\frac{1}{2}nn$	$\mu=3$; $A=\frac{1+\frac{1}{2}nn}{(1-nn^2)\sqrt{(1-nn)}}$
$\nu=3$; $A=1+\frac{3}{2}n^2$	$\mu=4$; $A=\frac{1+\frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}}$
$\nu=4$; $A=1+3n^2+\frac{3}{8}n^4$	$\mu=5$; $A=\frac{1+3nn+\frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}$

etc.

vnde concludimus, si fuerit

$$(1+n\,\text{cos.}\,\Phi)^{\nu}=A+B\,\text{cos.}\,\Phi+C\,\text{cos.}\,2\Phi+\text{etc.}$$
$$(1+n\,\text{cos.}\,\Phi)^{-\nu-1}=\mathfrak{A}+\mathfrak{B}\,\text{cos.}\,\Phi+\mathfrak{C}\,\text{cos.}\,2\Phi+\text{etc.}$$

fore $\mathfrak{A}=\frac{A}{(1-nn)^{\nu}\sqrt{(1-nn)}}$.

Quare cum pro casibus, quibus ν est numerus integer positiuus, valor ipsius A facile definiatur, etiam pro casibus, quibus est negatiuus, inde expedite assignabitur.

Scholion 2.

291. Cum pro casu $\mu=1$, supra valores singularum litterarum A, B, C, D etc. sint inuenti, scilicet posito breuitatis gratia $\frac{1-\sqrt{(1-nn)}}{n}=m$,

A =

$$A = \frac{1}{\sqrt{(1-nn)}};\ B = \frac{2m}{\sqrt{(1-nn)}};\ C = \frac{2mm}{\sqrt{(1-nn)}};\ D = \frac{2m^3}{\sqrt{(1-nn)}};$$

et in genere pro termino quocunque $N = \frac{2m^\lambda}{\sqrt{(1-nn)}}$: ſi pro ſimili termino caſu $\mu = 2$, ſcribamus N', erit $N' = \frac{\partial . Nn}{\partial n}$. Nunc autem eſt $\frac{\partial . Nn}{\partial n} = \frac{2m^\lambda}{(1-nn)^{\frac{3}{2}}} + \frac{2\lambda n m^{\lambda-1} \partial m}{\partial n \sqrt{(1-nn)}}$: tum vero $\frac{\partial m}{\partial n} = \frac{m}{n\sqrt{(1-nn)}}$, vnde colligimus

$$N' = \frac{2m^\lambda}{(1-nn)^{\frac{3}{2}}} + \frac{2\lambda m^\lambda}{1-nn} = \frac{2m^\lambda[1+\lambda\sqrt{(1-nn)}]}{(1-nn)\sqrt{(1-nn)}}.$$

Quare ſi ſtatuamus:

$$\frac{1}{(1+n\,\text{coſ.}\,\Phi)^2} = A + B\,\text{coſ.}\,\Phi + C\,\text{coſ.}\,2\Phi + D\,\text{coſ.}\,3\Phi + E\,\text{coſ.}\,4\Phi + \text{etc.}$$

erit

$$A = \frac{1}{(1-nn)^{\frac{3}{2}}};\ B = \frac{2m[1+\sqrt{(1-nn)}]}{(1-nn)^{\frac{3}{2}}};\ C = \frac{2m^2[1+2\sqrt{(1-nn)}]}{(1-nn)^{\frac{3}{2}}};$$

$$D = \frac{2m^3[1+3\sqrt{(1-nn)}]}{(1-nn)^{\frac{3}{2}}};\ \text{etc.}$$

Verum ſi exponens μ fuerit numerus fractus, coefficientes A, B, C, D, E, etc. haud aliter, ac per ſeries ſupra datas definiri poſſe videntur. Primus autem A modo peculiari vero proxime aſſignari poteſt, quemadmodum in problemate ſequente docemus.

Problema 34.

292. Pro euolutione formulae $(1 + n\,\text{coſ.}\,\Phi)^\nu$ in huiusmodi ſeriem $A + B\,\text{coſ.}\,\Phi + C\,\text{coſ.}\,2\Phi + D\,\text{coſ.}\,3\Phi + E\,\text{coſ.}\,4\Phi + \text{etc.}$ terminum abſolutum A vero proxime definire.

Solutio.

Cum necessario sit $n < 1$, series quidem supra inuenta pro A conuergit, verum si n parum ab vnitate deficiat, permultos terminos actu euolui oportet, antequam valor ipsius A satis exacte prodeat, praecipue si ν fuerit numerus mediocriter magnus tam positiuus quam negatiuus. Quoniam tamen posita euolutione huius formulae $(1 + n \cos. \Phi)^{-\nu-1} = \mathfrak{A} + \mathfrak{B} \cos. \Phi + \mathfrak{C} \cos. 2\Phi +$ etc. a termino $\mathfrak{A}$ illa A ita pendet, vt sit $A = (1 - nn)^{\nu + \frac{1}{2}} \mathfrak{A}$, pro hoc termino A inueniendo duplicem habemus seriem

$$A = 1 + \frac{\nu(\nu-1)}{2 \cdot 2} n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2 \cdot 2 \cdot 4 \cdot 4} n^4 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} n^6 + \text{etc.}$$

$$A = (1 - nn)^{\nu + \frac{1}{2}} \Big(1 + \frac{(\nu+1)(\nu+2)}{2 \cdot 2} n^2 + \frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)}{2 \cdot 2 \cdot 4 \cdot 4} n^4 + \frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)(\nu+5)(\nu+6)}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} n^6 + \text{etc.}\Big)$$

quouis casu ea vsurpari potest, quae magis conuergit. Verum tamen quia reliqui coefficientes B, C, D, E, etc. tandem conuergere debent, hinc alia via ad valorem ipsius A appropinquandi patet. Quoniam enim hi coefficientes alternatim per pares et impares potestates ipsius n definiuntur, sumto angulo quocunque α erit

$$(1 + n \cos. \alpha)^\nu = A + B \cos. \alpha + C \cos. 2\alpha + D \cos. 3\alpha + E \cos. 4\alpha + \text{etc. et}$$

$$(1 - n \cos. \alpha)^\nu = A - B \cos. \alpha + C \cos. 2\alpha - D \cos. 3\alpha + E \cos. 4\alpha - \text{etc.}$$

His igitur additis prodit

$$\tfrac{1}{2}(1 + n \cos. \alpha)^\nu + \tfrac{1}{2}(1 - n \cos. \alpha)^\nu = A + C \cos. 2\alpha + E \cos. 4\alpha + G \cos. 6\alpha + \text{etc.}$$

vbi si pro α scribamus $90° - \alpha$, erit

$\frac{1}{2}$(1

$$\tfrac{1}{2}(1 + n\,\text{fin.}\,\alpha)^\nu + \tfrac{1}{2}(1 - \text{fin.}\,\alpha)^\nu = \text{A} - \text{C}\,\text{cof.}\,2\alpha + \text{E}\,\text{cof.}\,4\alpha - \text{G}\,\text{cof.}\,6\alpha + \text{etc.}$$

vnde his additis, femiffis terminorum denuo tollitur. Formemus plures huiusmodi expreffiones, ac ponamus breuitatis gratia:

$$\tfrac{1}{4}(1+n\,\text{cof.}\alpha)^\nu+\tfrac{1}{4}(1-n\,\text{cof.}\alpha)^\nu+\tfrac{1}{4}(1+n\,\text{fin.}\alpha)^\nu+\tfrac{1}{4}(1-n\,\text{fin.}\alpha)^\nu=\mathfrak{A}$$
$$\tfrac{1}{4}(1+n\,\text{cof.}\beta)^\nu+\tfrac{1}{4}(1-n\,\text{cof.}\beta)^\nu+\tfrac{1}{4}(1+n\,\text{fin.}\beta)^\nu+\tfrac{1}{4}(1-n\,\text{fin.}\beta)^\nu=\mathfrak{B}$$
$$\tfrac{1}{4}(1+n\,\text{cof.}\gamma)^\nu+\tfrac{1}{4}(1-n\,\text{cof.}\gamma)^\nu+\tfrac{1}{4}(1+n\,\text{fin.}\gamma)^\nu+\tfrac{1}{4}(1-n\,\text{fin.}\gamma)^\nu=\mathfrak{C}$$

etc.

et pro coefficientibus B, C, D, E, etc. fcribamus refpectiue (1), (2), (3), (4), etc. quo facilius terminos ab initio quantumuis remotos repraefentare poffimus. Habebimus ergo

$$\mathfrak{A} = \text{A} + (4)\,\text{cof.}\,4\alpha + (8)\,\text{cof.}\,8\alpha + (12)\,\text{cof.}\,12\alpha + \text{etc.}$$
$$\mathfrak{B} = \text{A} + (4)\,\text{cof.}\,4\beta + (8)\,\text{cof.}\,8\beta + (12)\,\text{cof.}\,12\beta + \text{etc.}$$
$$\mathfrak{C} = \text{A} + (4)\,\text{cof.}\,4\gamma + (8)\,\text{cof.}\,8\gamma + (12)\,\text{cof.}\,12\gamma + \text{etc.}$$

etc.

Atque hinc fequentes approximationes adipifcimur.

I. Si capiamus $4\alpha = \frac{\pi}{2}$ feu $\alpha = \frac{\pi}{8}$, prodit

$$\mathfrak{A} = \text{A} - (8) + (16) - (24) + \text{etc.} \quad \text{Ergo}$$
$$\text{A} = \mathfrak{A} + (8) - (16) + (24) - \text{etc.}$$

Quare fi termini (8) et fequentes ob paruitatem contemni queant, erit fatis exacte $\text{A} = \mathfrak{A}$.

II. Sumamus duas feries ac ftatuamus $4\alpha = \frac{\pi}{4}$ et $4\beta = \frac{3\pi}{4}$ vt fit $\alpha = \frac{\pi}{16}$ et $\beta = \frac{3\pi}{16}$ erit $\text{cof.}\,4\alpha + \text{cof.}\,4\beta = 0$, $\text{cof.}\,8\alpha + \text{cof.}\,8\beta = 0$, $\text{cof.}\,12\alpha + \text{cof.}\,12\beta = 0$ et $\text{cof.}\,16\alpha + \text{cof.}\,\beta = -2$, vnde fequitur:

$$\mathfrak{A} + \mathfrak{B} = 2\text{A} - 2(16) + 2(32) - 2(48) + \text{etc.}$$

 ideo-

ideoque

$$A = \tfrac{1}{2}(\mathfrak{A} + \mathfrak{B}) + (16) - (32) + \text{ etc.}$$

vbi numeri (16), (32) plerumque tam erunt parvi, vt negligi queant.

III. Addamus tres ſeries, ac ſtatuamus $4\alpha = \frac{\pi}{6}$; $4\beta = \frac{3\pi}{6}$; $4\gamma = \frac{5\pi}{6}$; vt ſit $\alpha = \frac{\pi}{24}$; $\beta = \frac{\pi}{8}$; $\gamma = \frac{5\pi}{24}$; eritque

$\operatorname{cof.} 4\alpha + \operatorname{cof.} 4\beta + \operatorname{cof.} 4\gamma = 0$	$\operatorname{cof.} 16\alpha + \operatorname{cof.} 16\beta + \operatorname{cof.} 16\gamma = 0$
$\operatorname{cof.} 8\alpha + \operatorname{cof.} 8\beta + \operatorname{cof.} 8\gamma = 0$	$\operatorname{cof.} 20\alpha + \operatorname{cof.} 20\beta + \operatorname{cof.} 20\gamma = 0$
$\operatorname{cof.} 12\alpha + \operatorname{cof.} 12\beta + \operatorname{cof.} 12\gamma = 0$	$\operatorname{cof.} 24\alpha + \operatorname{cof.} 24\beta + \operatorname{cof.} 24\gamma = -3$:

vnde colligitur

$$A = \tfrac{1}{3}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}) + (24) - (48) + \text{ etc.}$$

IV. Si haec determinatio non ſatis exacta videatur, addantur quatuor eiusmodi expreſſiones $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, ſitque

$$4\alpha = \tfrac{\pi}{8};\ 4\beta = \tfrac{3\pi}{8};\ 4\gamma = \tfrac{5\pi}{8};\ 4\delta = \tfrac{7\pi}{8};$$

ac reperietur

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} = 4A - 4(32) + 4(64) - \text{ etc.}$$

ergo multo propius

$$A = \tfrac{1}{4}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D}).$$

Corollarium 1.

293. Ex inuento autem valore A ſequens B ſatis expedite reperitur, cum ſit

$$B = \frac{2(\nu+2)}{n} \int A n \partial n - 2An.$$

Quatenus ergo in A ingreditur membrum $(1 \pm n \operatorname{cof.} \alpha)^\nu$, vel $(1 + nf)^\nu$, dum f omnes illos ſinus et coſinus complectitur, inde pro B oritur

$$\frac{2(\nu+2)}{n} \int n \partial n (1+nf)^\nu - 2n(1+nf)^\nu = \frac{2 - 2(1-\nu nf)(1+nf)^\nu}{(\nu+1)nff}$$

Corol-

Corollarium 2.

294. Cognitis autem coefficientibus A et B, quemadmodum sequentes omnes ex illis deriuari possint, supra ostendimus. Iis vero inuentis integratio formulae $\partial\Phi(1+n\,\text{cos}.\Phi)^\nu$ per se est manifesta.

Problema 35.

295. Integrale formulae $\partial\Phi\, l(1+n\,\text{cos}.\Phi)$ per seriem secundum sinus angulorum Φ, 2Φ, 3Φ, etc. progredientem euoluere.

Solutio.

Cum sit

$$l(1+n\,\text{cos}.\Phi) = n\,\text{cos}.\Phi - \tfrac{1}{2}n^2\,\text{cos}.\Phi^2 + \tfrac{1}{3}n^3\,\text{cos}.\Phi^3 - \tfrac{1}{4}n^4\,\text{cos}.\Phi^4 + \text{etc.}$$

erit his potestatibus ad simplices cosinus reductis.

$$\begin{array}{llllll} l(1+n\,\text{cos}.\Phi) = & & +n\,\text{cos}.\Phi & -\tfrac{1}{2}.\tfrac{1}{2}nn\,\text{cos}.2\Phi & +\tfrac{1}{3}.\tfrac{1}{4}n^3\,\text{cos}.3\Phi & -\tfrac{1}{4}.\tfrac{1}{8}n^4\,\text{cos}.4\Phi \\ & -\tfrac{1}{2}.\tfrac{1}{2}n^2 & +\tfrac{1}{3}.\tfrac{3}{4}n^3 & -\tfrac{1}{4}.\tfrac{4}{8}n^4 & +\tfrac{1}{5}.\tfrac{5}{16}n^5 & \\ & -\tfrac{1}{4}.\tfrac{3}{8}n^4 & +\tfrac{1}{5}.\tfrac{10}{16}n^5 & -\tfrac{1}{6}.\tfrac{15}{32}n^6 & & \\ & -\tfrac{1}{6}.\tfrac{10}{32}n^6 & +\tfrac{1}{7}.\tfrac{35}{64}n^7 & & & \\ & -\tfrac{1}{8}.\tfrac{35}{128}n^8. & & & & \end{array}$$

Quare si ponamus

$$l(1+n\,\text{cos}.\Phi) = -A + B\,\text{cos}.\Phi - C\,\text{cos}.2\Phi + D\,\text{cos}.3\Phi - \text{etc.}$$

erit

$$A = +\tfrac{1}{2}.\frac{n^2}{2} + \frac{1.3}{2.2}.\frac{n^4}{4} + \frac{1.3.5}{2.4.6}.\frac{n^6}{6} + \frac{1.3.5.7}{2.4.6.8}.\frac{n^8}{8} + \text{etc.}$$

considerato ergo numero n vt variabili, erit

$$\frac{n\,\partial A}{\partial n} = \tfrac{1}{2}nn + \frac{1.3}{2.4}n^4 + \frac{1.3.5}{2.4.6}n^6 + \text{etc.} = \frac{1}{\sqrt{(1-nn)}} - 1.$$

Hinc $\partial A = \frac{\partial n}{n\sqrt{(1-nn)}} - \frac{\partial n}{n}$, vnde integratio praebet;

$$A = l\,\frac{1-\sqrt{(1-nn)}}{n} - l\,n + C = l\,\frac{2-2\sqrt{(1-nn)}}{nn}.$$

hoc

hoc enim modo euanescente n, fit $A = l\,1 = 0$. Tum vero erit

$$\tfrac{1}{2}B = \tfrac{1}{2}\,.\,n + \tfrac{1.3}{2.4}\,.\,\tfrac{n^3}{3} + \tfrac{1.3.5}{2.4.6}\,.\,\tfrac{n^5}{5} + \text{etc.}$$

vnde differentiatio praebet

$$\tfrac{nn\,\partial B}{2\,\partial n} = \tfrac{1}{2}nn + \tfrac{1.3}{2.4}n^4 + \tfrac{1.3.5}{2.4.6}n^6 + \text{etc.} = \tfrac{1}{\sqrt{(1-nn)}} - 1:$$

ergo $\tfrac{1}{2}\partial B = \frac{\partial n}{nn\sqrt{(1-nn)}} - \frac{\partial n}{nn}$, et integrando

$$\tfrac{1}{2}B = \tfrac{-\sqrt{(1-nn)}}{n} + \tfrac{1}{n} + C = \tfrac{1-\sqrt{(1-nn)}}{n}$$

integrali ita determinato vt euanescat posito $n = 0$.

Quocirca pro binis primis terminis habemus:

$$A = l\,\tfrac{2-2\sqrt{(1-nn)}}{nn} \quad \text{et} \quad B = \tfrac{2-2\sqrt{(1-nn)}}{n};$$

vt sit $A = l\,\frac{B}{n}$. At pro reliquis differentiemus aequationem assumtam

$$\tfrac{-n\,\partial\Phi\,\sin.\Phi}{1+n\cos.\Phi} = -B\,\partial\Phi\sin.\Phi + 2C\,\partial\Phi\sin.2\Phi$$
$$-3D\,\partial\Phi\sin.3\Phi + 4E\,\partial\Phi\sin.4\Phi - \text{etc.}$$

seu

$$0 = \tfrac{n\sin.\Phi}{1+n\cos.\Phi} - B\sin.\Phi + 2C\sin.2\Phi$$
$$-3D\sin.3\Phi + 4E\sin.4\Phi - \text{etc.}$$

Quare per $2 + 2n\cos.\Phi$ multiplicando prodit:

$$\begin{array}{llllll} 0 = 2n\sin.\Phi & -2B\sin.\Phi & +4C\sin.2\Phi & -6D\sin.3\Phi & +8E\sin.4\Phi & -\text{etc.} \\ & -Bn & +2Cn & -3Dn & & \\ & +2Cn & -3Dn & +4En & -5Fn & \end{array}$$

vnde colligimus:

$$C = \tfrac{B-n}{n};\; D = \tfrac{4C-Bn}{3n};\; E = \tfrac{6D-2Cn}{4n};\; F = \tfrac{8E-3Dn}{5n};$$

Cum igitur sit $B = \frac{2-2\sqrt{(1-nn)}}{n}$, erit $C = \frac{2-nn-\sqrt{(1-nn)}}{nn}$, seu $C = \left(\frac{1-\sqrt{(1-nn)}}{n}\right)^2$: tum vero

D =

$D = \frac{2}{3}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^3$; $E = \frac{2}{4}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^4$; $F = \frac{2}{5}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^5$; etc.

Hinc si breuitatis gratia ponamus $\frac{1-\sqrt{(1-nn)}}{n} = m$, erit

$$l(1+n\,\mathrm{cos}.\,\Phi) = -l\frac{2m}{n} + \frac{2}{1}m\,\mathrm{cos}.\,\Phi - \frac{2}{2}m^2\,\mathrm{cos}.\,2\Phi + \frac{2}{3}m^3\,\mathrm{cos}.\,3\Phi - \frac{2}{4}m^4\,\mathrm{cos}.\,4\Phi + \text{etc.}$$

ideoque integrale quaesitum:

$$\int \partial\Phi\, l(1+n\,\mathrm{cos}.\,\Phi) = \text{Const.} - \Phi\, l\frac{2m}{n} + \frac{2}{1}m\,\mathrm{fin}.\,\Phi - \frac{2}{4}m^2\,\mathrm{fin}.\,2\Phi + \frac{2}{9}m^3\,\mathrm{fin}.\,3\Phi - \frac{2}{16}m^4\,\mathrm{fin}.\,4\Phi + \frac{2}{25}m^5\,\mathrm{fin}.\,5\Phi - \text{etc.}$$

Corollarium.

296. Quodsi ergo ponamus $n = 1$, erit $m = 1$ et

$$l(1+\mathrm{cof}.\Phi) = -l\,2 + \frac{2}{1}\mathrm{cof}.\Phi - \frac{2}{2}\mathrm{cof}.2\Phi + \frac{2}{3}\mathrm{cof}.3\Phi - \frac{2}{4}\mathrm{cof}.4\Phi + \text{etc.}$$

et

$$l(1-\mathrm{cof}.\Phi) = -l\,2 - \frac{2}{1}\mathrm{cof}.\Phi - \frac{2}{2}\mathrm{cof}.2\Phi - \frac{2}{3}\mathrm{cof}.3\Phi - \frac{2}{4}\mathrm{cof}.4\Phi - \text{etc.}$$

Cum iam sit

$1 + \mathrm{cof}.\,\Phi = 2\,\mathrm{cof}.\,\frac{1}{2}\Phi^2$ et $1 - \mathrm{cof}.\,\Phi = 2\,\mathrm{fin}.\,\frac{1}{2}\Phi^2$, erit

$l\,\mathrm{cof}.\frac{1}{2}\Phi = -l\,2 + \mathrm{cof}.\Phi - \frac{1}{2}\mathrm{cof}.2\Phi + \frac{1}{3}\mathrm{cof}.3\Phi - \frac{1}{4}\mathrm{cof}.4\Phi + \text{etc.}$ et

$l\,\mathrm{fin}.\frac{1}{2}\Phi = -l\,2 - \mathrm{cof}.\Phi - \frac{1}{2}\mathrm{cof}.2\Phi - \frac{1}{3}\mathrm{cof}.3\Phi - \frac{1}{4}\mathrm{cof}.4\Phi - \text{etc.}$

hinc

$$l\,\mathrm{tang}.\frac{1}{2}\Phi = -2\,\mathrm{cof}.\Phi - \frac{2}{3}\mathrm{cof}.3\Phi - \frac{2}{5}\mathrm{cof}.5\Phi - \frac{2}{7}\mathrm{cof}.7\Phi - \text{etc.}$$

CAPVT VII.

METHODVS GENERALIS

INTEGRALIA QVAECVNQVE PROXIME INVENIENDI.

Problema 36.

297.

Formulae integralis cuiuscunque $y = \int X \partial x$ valorem vero proxime indagare.

Solutio.

Cum omnis formula integralis per se sit indeterminata, ea semper ita determinari solet, vt si variabili x certus quidam valor, puta a, tribuatur, ipsum integrale $y = \int X \partial x$ datum valorem, puta b, obtineat. Integratione igitur hoc modo determinata, quaestio huc redit, si variabili x alius quicunque valor ab a diuersus tribuatur, valor, quem tum integrale y sit habiturum, definiatur. Tribuamus ergo ipsi x primo valorem parum ab a discrepantem, puta $x = a + \alpha$, vt α sit quantitas valde parua: et quia functio X parum variatur, siue pro x scribatur a siue $a + \alpha$, eam tanquam constantem spectare licebit. Hinc ergo formulae differentialis $X \partial x$ integrale erit $Xx + \text{Const.} = y$; sed quia posito $x = a$, fieri debet $y = b$, et valor ipsius X quasi manet immutatus, erit $Xa + \text{Const.} = b$, ideoque $\text{Const.} = b - Xa$, vnde consequimur $y = b + X(x - a)$. Quare si ipsi x valorem $a + \alpha$ tribuamus, habebimus valorem conuenientem ipsius y, qui sit $= b + \beta$; ac iam simili modo ex hoc casu definire poterimus y, si ipsi x tribuatur alius valor parum superans $a + \alpha$: posito igitur $a + \alpha$ loco x, valor ipsius

ipsius X inde ortus denuo pro constante haberi poterit, indeque fiet $y = b + \beta + X(x - a - \alpha)$. Hanc igitur operationem continuare licet quousque lubuerit, cuius ratio quo melius perspiciatur, rem ita repraesentemus:

$$
\begin{array}{llll}
\text{si } x = a & \text{fiat } X = A & \text{et } y = b & \\
\text{si } x = a' & . . X = A' & . . y = b' & = b + A(a' - a) \\
\text{si } x = a'' & . . X = A'' & . . y = b'' & = b' + A'(a'' - a') \\
\text{si } x = a''' & . . X = A''' & . . y = b''' & = b'' + A''(a''' - a'') \\
& & \text{etc.} &
\end{array}
$$

vbi valores a, a', a'', a''', etc. secundum differentias valde paruas procedere ponuntur. Erit ergo $b' = b + A(a' - a)$, quippe in quam abit formula inuenta $y = b + X(x - a)$: fit enim $X = A$, quia ponitur $x = a$, tum vero tribuitur ipsi x valor $= a'$, cui respondet $y = b'$: simili modo erit $b'' = b' + A'(a'' - a')$; tum $b''' = b'' + A''(a''' - a'')$ etc. vti supra posuimus. Restituendo ergo valores praecedentes habebimus:

$$
\begin{aligned}
b' &= b + A(a' - a) \\
b'' &= b + A(a' - a) + A'(a'' - a') \\
b''' &= b + A(a' - a) + A'(a'' - a') + A''(a''' - a'') \\
b'''' &= b + A(a' - a) + A'(a'' - a') + A''(a''' - a'') + A'''(a'''' - a''') \\
& \text{etc.}
\end{aligned}
$$

vnde si x quantumuis excedet a, series a', a'', a''', etc. crescendo continuetur ad x, et vltimum aggregatum dabit valorem ipsius y.

Corollarium I.

298. Si incrementa, quibus x augetur, aequalia statuantur scilicet $= \alpha$, vt sit $a' = a + \alpha$, $a'' = a + 2\alpha$, $a''' = a + 3\alpha$, etc. quibus valoribus pro x substitutis functio X abe-

at in A′, A″, A‴, etc. atque vltimus illorum, puta $a+n\alpha$, fit $=x$, horum vero X, erit

$$y=b+\alpha(A+A'+A''+A''' \ldots\ldots +X).$$

Corollarium 2.

299. Valor ergo integralis y per summationem seriei A, A′, A″ .... X, cuius termini ex formula X formantur, ponendo loco x successiue a, $a+\alpha$, $a+2\alpha$ .... $a+n\alpha$, eruitur. Summa enim illius seriei per differentiam α multiplicata et ad b adiecta, dabit valorem ipsius y, qui ipsi $x=a+n\alpha$ respondet.

Corollarium 3.

300. Quo minores statuuntur differentiae, secundum quas valor ipsius x increscat, eo accuratius hoc modo valor ipsius y definitur. Siquidem termini seriei A, A′, A″, etc. inde etiam secundum paruas differentias progrediantur, nisi enim hoc eueniat, illa determinatio nimis erit incerta.

Corollarium 4.

301. Haec ergo approximatio ex doctrina serierum ita explicatur:

Ex indicibus a, a', a'', a''' . . . . . x formetur
series A, A′, A′, A‴ X

cuius ergo terminus generalis X ex formula differentiali $\partial y = X\partial x$ datur. Tum in hac serie sit terminus vltimum praecedens ′X, respondens indici ′x; hincque noua formetur series

$$A(a'-a);\ A'(a''-a');\ A''(a'''-a'') \ldots\ldots {}'X(x-{}'x),$$

cuius summa si ponatur $=S$, erit integrale $y=\int X\partial x=b+S$, proxime.

Scho-

Scholion 1.

302. Hoc modo integratio vulgo explicari folet, vt dicatur, effe fummatio omnium valorum formulae differentialis $X \partial x$, fi variabil x fucceffiue omnes valores a dato quodam a vsque ad x tribuantur, qui fecundum differentiam ∂x procedunt, hanc differentiam autem infinite paruam accipi oportere. Similis igitur haec ratio integrationem repraefentandi eft illi, qua in Geometria lineae vt aggregata infinitorum punctorum concipi folent, quae idea, quemadmodum fi rite explicetur, admitti poteft, ita etiam illi integrationis explicatio tolerari poteft, dummodo ad vera principia, vti hic fecimus, reuocetur, vt omni cauillationi occurratur. Ex methodo igitur expofita vtique patet, integrationem per fummationem vero proxime obtineri poffe, neque vero exacte expediri, nifi differentiae infinite paruae, hoc eft nullae, ftatuantur. Atque ex hoc fonte tam nomen integrationis, quae etiam fummatio vocari folet, quam fignum integralis $\int$ eft natum, quae, re bene explicata, omnino retineri poffunt.

Scholion 2.

303. Si pro fingulis interuallis, in quae faltum ab a ad x diftinximus, quantitates A, A', A'', A''', etc. reuera effent conftantes, integrale $\int X \partial x$ accurate impetraremus. Eatenus ergo error ineft, quatenus pro fingulis illis interuallis iftae quantitates non funt conftantes. Ac pro primo quidem interuallo, quo variabilis x a termino a ad a' procedit, A eft valor ipfius X termino a conueniens, alteri autem termino a' refpondet A'; vnde quatenus non eft $A' = A$, eatenus error irrepit: cum igitur in iftius interualli initio fit $X = A$, in fine autem $X = A'$, conueniret potius medium quoddam inter A et A' affumi, id quod in correctione huius methodi mox tradenda obferuabitur. Interim hic notaffe iuuabit, pari iure pro

 quo-

quouis interuallo valorem tam finalem quam initialem capi posse, vbi simul hoc perspicitur, si altero modo in excessu peccetur, altero plerumque in defectu errari. Ex quo hinc binas expressiones eruere licet, quarum altera valorem ipsius y nimis magnum, altera nimis paruum sit praebitura, ita vt illae quasi limites veri valoris ipsius y constituant. Quemadmodum ergo rem repraesentauimus §. 301. valor ipsius $y = \int X \partial x$ intra hos duos limites continebitur

$$b + A(a'-a) + A'(a''-a') + A''(a'''-a'') \ldots + {}'X(x-{}'x) \text{ et}$$
$$b + A'(a'-a) + A''(a''-a') + A'''(a'''-a'') \ldots + X(x-{}'x)$$

quibus cognitis, ad veritatem propius accedere licet.

Scholion 3.

304. Iam notauimus interualla illa, per quae x successiue increscere assumimus, ideo valde parua statui debere, vt valores respondentes A, A′, A″, etc. parum a se inuicem discrepent: atque hinc potissimum iudicari oportet, vtrum illa interualla $a' - a$, $a'' - a'$, $a''' - a''$, etc. inter se aequalia an inaequalia capi conueniat. Vbi enim valor ipsius X, mutando x, parum mutatur, ibi interualla, per quae x procedit, tuto maiora constitui possunt; vbi autem euenit, vt ipsi x leui mutatione inducta, functio X vehementer varietur, ibi interualla minima accipi debent. Veluti si sit $X = \frac{1}{\sqrt{(1-xx)}}$, perspicuum est, vbi x proxime ad vnitatem accedit, quantumuis paruum interuallum, per quod x augeatur, accipiatur, functionem X maximam mutationem pati posse, quia tandem sumto $x = 1$, ea adeo in infinitum excrescit. His igitur casibus ista approximatione pro eo saltem interuallo, in cuius altero termino X fit infinita, vti non licet; sed huic incommodo facile remedium affertur, dum formula ope idoneae substitutionis in aliam transformatur, vel dum pro hoc saltem interuallo peculiaris

culiaris integratio instituitur. Veluti si proposita sit formula $\frac{x\partial x}{\sqrt{(1-x^3)}}$, pro interuallo ab $x = 1 - \omega$ ad $x = 1$ illa methodo integrale non reperitur: at posito $x = 1 - z$, quia termini ipsius z sunt o et ω, erit z quantitas minima, vnde formula erit $\frac{\partial z(1-z)}{\sqrt{(3z - 3z^2 + z^3)}} = \frac{\partial z}{\sqrt{3z}}$, cuius integrale $\frac{2\sqrt{z}}{\sqrt{3}}$ pro interuallo illo praebet partem integralis $\frac{2\sqrt{\omega}}{\sqrt{3}}$. Quod artificium in omnibus huiusmodi casibus adhiberi potest; ipsam autem methodum descriptam aliquot exemplis illastrari opus est.

Exemplum 1.

305. *Integrale $y = \int x^n \partial x$ ita sumtum, vt euanescat posito $x = 0$, proxime exhibere.*

Hic est $a = 0$ et $b = 0$, tum $X = x^n$, iam valores ipsius x a o crescant per communem differentiam α, vt sint

indices $0, \alpha, 2\alpha, 3\alpha, 4\alpha \ldots\ldots x$
series $0, \alpha^n, 2^n\alpha^n, 3^n\alpha^n, 4^n\alpha^n \ldots\ldots x^n$

et terminus vltimum praecedens est $(x-\alpha)^n$, quare integralis $y = \int x^n \partial x = \frac{1}{n+1} x^{n+1}$ limites sunt

$$\alpha[0 + \alpha^n + 2^n\alpha^n + 3^n\alpha^n + \ldots\ldots + (x-\alpha)^n] \text{ et}$$
$$\alpha(\alpha^n + 2^n\alpha^n + 3^n\alpha^n + \ldots\ldots\ldots + x^n)$$

qui eo erunt arctiores, quo minus interuallum α accipiatur. Ita si $\alpha = 1$, erunt limites:

$$0 + 1 + 2^n + 3^n + 4^n \ldots\ldots + (x-1)^n \text{ et}$$
$$1 + 2^n + 3^n + 4^n + \ldots\ldots\ldots + x^n,$$

si sumatur $\alpha = \frac{1}{2}$, erunt limites:

$$\frac{1}{2^{n+1}}[0 + 1 + 2^n + 3^n + 4^n + \ldots\ldots + (2x-1)^n] \text{ et}$$
$$\frac{1}{2^{n+1}}[1 + 2^n + 3^n + 4^n + \ldots\ldots\ldots + (2x)^n];$$

ac

ac si in genere sit $\alpha = \frac{1}{m}$, erunt limites:

$$\frac{1}{m^{n+1}}[0 + 1 + 2^n + 3^n + 4^n + \ldots\ldots + (mx-1)^n] \text{ et}$$

$$\frac{1}{m^{n+1}}[1 + 2^n + 3^n + 4^n + \ldots\ldots\ldots + (mx)^n],$$

quorum hic illum superat excessu $\frac{x^n}{m}$; vnde patet si numerus m sumatur infinitus, vtrumque limitem verum integralis $y = \frac{1}{n+1}x^{n+1}$ esse praebiturum valorem.

Corollarium 1.

306. Serici ergo $1 + 2^n + 3^n + 4^n + \ldots + (mx)^n$ summa eo propius ad $\frac{1}{n+1}(mx)^{n+1}$ accedit, quo maior capiatur numerus m; quare posito $mx = z$, huius progressionis

$$1 + 2^n + 3^n + 4^n + \ldots\ldots + z^n,$$

summa eo propius ad $\frac{1}{n+1}z^{n+1}$ accedit, quo maior fuerit numerus z.

Corollarium 2.

307. Ex priore autem limite posito $mx = z$, eadem quantitas $\frac{1}{n+1}z^{n+1}$ proxime exhibet summam huius seriei

$$0 + 1 + 2^n + 3^n + 4^n + \ldots\ldots + (z-1)^n,$$

vnde medium sumendo erit accuratius:

$$1 + 2^n + 3^n + 4^n \ldots\ldots + (z-1)^n + \tfrac{1}{2}z^n = \frac{1}{n+1}z^{n+1}$$

seu addendo vtrinque $\frac{1}{2}z^n$, habebimus

$$1 + 2^n + 3^n + 4^n \ldots\ldots + z^n = \frac{1}{n+1}z^{n+1} + \tfrac{1}{2}z^n,$$

proxime quod congruit cum iis, quae de vera huius progressionis summa sunt cognita.

Exem-

Exemplum 2.

308. *Integrale* $y = \int \frac{\partial x}{x^n}$ *ita sumtum, vt euanescat posito* $x = 1$, *proxime exhibere.*

Erit ergo $a = 1$ et $b = 0$, vnde si ab a ad x intervallum progressionis statuatur $= \alpha$, erunt indices

$$a,\ a + \alpha,\ a + 2\alpha,\ a + 3\alpha,\ \ldots\ldots\ x,$$

et termini seriei

$$\frac{1}{a^n},\ \frac{1}{(a+\alpha)^n},\ \frac{1}{(a+2\alpha)^n},\ \frac{1}{(a+3\alpha)^n},\ \ldots\ \frac{1}{x^n},\ = \mathrm{X},$$

vbi terminus vltimum praecedens est $\frac{1}{(x-\alpha)^n} = \mathrm{X}'$. Cum nunc nostrum integrale sit $y = \frac{1}{n-1} - \frac{1}{(n-1)x^{n-1}}$, eius valor intra hos limites continebitur:

$$\alpha\left[1 + \frac{1}{(1+\alpha)^n} + \frac{1}{(1+2\alpha)^n} + \frac{1}{(1+3\alpha)^n} + \ldots + \frac{1}{(x-\alpha)^n}\right] \text{ et}$$

$$\alpha\left[\frac{1}{(1+\alpha)^n} + \frac{1}{(1+2\alpha)^n} + \frac{1}{(1+3\alpha)^n} + \ldots\ldots + \frac{1}{x^n}\right].$$

Quare posito $\alpha = \frac{1}{m}$, erunt hi limites:

$$m^{n-1}\left[\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \ldots + \frac{1}{(mx-1)^n}\right] \text{ et}$$

$$m^{n-1}\left[\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \frac{1}{(m+4)^n} + \ldots + \frac{1}{(mx)^n}\right]$$

qui, quo maior accipiatur numerus m, eo propius ad valorem integralis $\frac{1}{n-1} - \frac{1}{(n-1)x^{n-1}}$ accedunt. Notandum autem est, casu $n = 1$ integrale fore $= lx$.

Corollarium 1.

309. Quodsi ponamus $mx = m + z$, vt sit $x = \frac{m+z}{m}$, prodibunt hae progressiones:

$$m^{n-1}\left(\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \dots + \frac{1}{(m+z-1)^n}\right) \text{ et}$$

$$m^{n-1}\left(\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \dots + \frac{1}{(m+z)^n}\right)$$

quarum summa alterius maior est, alterius minor quam

$$\frac{1}{n-1} - \frac{m^{n-1}}{(n-1)(m+z)^{n-1}} = \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)(m+z)^{n-1}}:$$

casu autem $n = 1$, haec expressio abit in $l\left(1 + \frac{z}{m}\right)$.

Corollarium 2.

310. Cum prior progressio maior sit quam posterior, erit

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \dots$$
$$\dots + \frac{1}{(m+z-1)^n} > \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}}$$

$$\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \dots$$
$$\dots + \frac{1}{(m+z)^n} < \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}}:$$

addatur hic vtrinque $\frac{1}{m^n}$, ibi vero $\frac{1}{(m+z)^n}$, et sumatur medium arithmeticum, erit exactius

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\ldots\ldots+\frac{1}{(m+z)^n}$$
$$=\frac{(2m+n-1)(m+z)^n-(2z+2m-n+1)m^n}{2(n-1)m^n(m+z)^n},$$

quae expressio casu $n=1$, abit in $l(1+\frac{z}{m})+\frac{1}{2m}+\frac{1}{2(m+z)}$.

Corollarium 3.

311. Ponatur $z=mv$, et habebimus sequentis seriei summam proxime expressam:

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\ldots\ldots+\frac{1}{m^n(1+v)^n}$$
$$=\frac{(2m+n-1)(1+v)^n-2m(1+v)+n-1}{2(n-1)m^n(1+v)^n},$$

et casu $n=1$

$$\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{m+mv}=l(1+v)+\frac{2+v}{2m(1+v)}:$$

vnde si $v=1$, erit proxime

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\ldots+\frac{1}{2^n m^n}$$
$$=\frac{2^n(2m+n-1)-4m+n-1}{2^{n+1}(n-1)m^n},\ \text{et}$$

$$\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\ldots\ldots+\frac{1}{2m}=l\,2+\frac{3}{4m}.$$

Corollarium 4.

312. Hinc nascitur regula, logarithmos quantumuis magnorum numerorum proxime assignandi, dum series vulgares tantum pro numeris parum ab vnitate differentibus, valent. Scribamus enim u pro $1+v$, et habebimus

$$l\,u=\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\ldots\ldots+\frac{1}{mu}-\frac{1-u}{2mu}:$$

vnde lu eo accuratius definitur, quo maior ſumatur numerus m.

Exemplum 3.

313. *Integrale* $y = \int \frac{c\,\partial x}{cc+xx}$ *ita ſumtum, vt euaneſcat poſito* $x = 0$, *proxime exprimere.*

Hoc integrale vt nouimus, eſt $y =$ Ang. tang. $\frac{x}{c}$, ad quem valorem proxime exhibendum, eſt $a = 0$, et $b = 0$; ſi ergo valor ipſius x ab 0 per differentiam conſtantem α creſcere ſtatuatur, ob $X = \frac{c}{cc+xx}$, erunt eius valores

pro indicibus $0 \quad \alpha \quad 2\alpha \quad \ldots\ldots \quad x$;

$\frac{1}{c}; \quad \frac{c}{cc+\alpha\alpha}; \quad \frac{c}{cc+4\alpha\alpha}; \ldots\ldots \frac{c}{cc+xx};$

cuius terminus vltimum praecedens eſt $X = \frac{c}{cc+(x-\alpha)^2}$.

Quare integralis noſtri $y =$ Ang. tang. $\frac{x}{c}$ valor proxime eſt

$$\alpha\left(\frac{1}{c} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \ldots\ldots + \frac{c}{cc+(x-\alpha)^2}\right);$$

alter vero proxime minor, quia hic eſt nimis magnus, eſt

$$\alpha\left(\frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \ldots\ldots + \frac{c}{cc+xx}\right).$$

Inter quos ſi medium capiatur, ibi $\alpha.\frac{1}{c}$, hic vero $\alpha.\frac{c}{cc+xx}$ adiiciendo, propius erit

$$\alpha\left(\frac{c}{cc} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \ldots\ldots + \frac{c}{cc+xx}\right)$$
$$= \text{Ang. tang.}\,\frac{x}{c} + \frac{\alpha}{2}\left(\frac{1}{c} + \frac{c}{cc+xx}\right)$$
$$= \text{Ang. tang.}\,\frac{x}{c} + \frac{\alpha(2cc+xx)}{2c(cc+xx)}.$$

Pro hoc ergo angulo valorem proxime verum habemus

$$\text{Ang. tang.}\,\frac{x}{c} = \alpha c\left(\frac{1}{cc} + \frac{1}{cc+\alpha\alpha} + \frac{1}{cc+4\alpha\alpha} + \ldots\ldots + \frac{1}{cc+xx}\right) - \frac{\alpha(2cc+xx)}{2c(cc+xx)},$$

qui eo minus a veritate diſcrepabit, quo minor fuerit α numerus

merus ratione ipsius c. Quodsi ergo pro c numerum valde magnum sumamus, pro α vnitatem accipere licet; vnde posito $x = cv$, erit

$$\text{Ang. tang. } v = c\left(\frac{1}{cc} + \frac{1}{cc+1} + \frac{1}{cc+4} + \frac{1}{cc+9} \ldots\ldots + \frac{1}{cc+ccvv}\right) - \frac{(2+vv)}{2c(1+vv)},$$

idque eo exactius, quo maior capiatur numerus c.

Corollarium 1.

314. Si ponamus $c = 1$, quo casu error insignis esse debet, fiet

$$\text{Ang. tang. } v = 1 + \frac{1}{1+1} + \frac{1}{1+4} + \frac{1}{1+9} + \ldots\ldots + \frac{1}{1+vv} - \frac{(2+vv)}{2(1+vv)}.$$

Sit $v = 1$, erit Ang. tang. $1 = \frac{\pi}{4} = 1 + \frac{1}{2} - \frac{3}{4} = \frac{3}{4}$, hincque $\pi = 3$, quod non multum abhorret a vero; si ponamus $c = 2$, prodit

$$\text{Ang. tang. } v = 2\left(\frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+4} + \frac{1}{4+9} + \ldots\ldots + \frac{1}{4+4vv}\right) - \frac{(2+vv)}{4(1+vv)},$$

vnde si $v = 1$, colligitur

$$\text{Ang. tang. } 1 = \frac{\pi}{4} = 2\left(\frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+4}\right) - \frac{3}{8} = \frac{23}{20} - \frac{3}{8} = \frac{31}{40},$$

sicque $\pi = \frac{31}{10} = 3, 1$, propius accedens.

Corollarium 2.

315. Sit $c = 6$, eritque

$$\text{Ang. tang. } v = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4} + \ldots\ldots + \frac{1}{36+36vv}\right) - \frac{(2+vv)}{12(1+vv)},$$

vnde si $v = \frac{1}{2}$ et $v = \frac{1}{3}$, oritur:

$$\text{Ang. tang. } \tfrac{1}{2} = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4} + \frac{1}{36+9}\right) - \frac{3}{20},$$

$$\text{Ang. tang. } \tfrac{1}{3} = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4}\right) - \frac{19}{120}.$$

At est Ang. tang. $\frac{1}{2}$ + Ang. tang. $\frac{1}{3}$ = Ang. tang. $1 = \frac{\pi}{4}$. Ergo

$$\frac{\pi}{4} = 12\left(\frac{1}{36} + \frac{1}{37} + \frac{1}{40}\right) + \frac{2}{15} - \frac{37}{120} = \frac{1063}{1110} - \frac{7}{40} = \frac{695}{888},$$

seu $\pi = \frac{695}{222} = 3, 1306$.

Corol-

Corollarium 3.

316. Sin autem ibi ſtatim ponamus $v = 1$, erit

$$\frac{\pi}{4} = 6\left(\frac{1}{36} + \frac{1}{37} + \frac{1}{40} + \frac{1}{45} + \frac{1}{52} + \frac{1}{61} + \frac{1}{72}\right) - \frac{1}{8},$$

vnde fit $\pi = 3,13696$ multo propius veritati; plurium ſcilicet terminorum additio propius ad veritatem perducit.

Problema 37.

317. Methodum ad integralium valores appropinquandi ante expoſitam, perfectiorem reddere, vt minus a veritate aberretur.

Solutio.

Sit $y = \int X \partial x$ formula integralis propoſita, cuius valorem iam conſtet eſſe $y = b$, ſi ponatur $x = a$, ſiue is ſit datus per ipſam integrationis conditionem, ſiue iam per aliquot operationes inde deriuatus; ac tribuamus iam ipſi x valorem parum ſuperantem illum a, cui reſpondet $y = b$, tum vero fiat $X = A$, ſi ponatur $x = a$. In ſuperiori autem methodo aſſumſimus, dum x parum ſupra a excreſcit, manere X conſtantem $= A$, ideoque fore $\int X \partial x = A(x - a)$. At quatenus X non eſt conſtans, eatenus non eſt $\int X \partial x = X \times (x - a)$, ſed reuera habetur $\int X \partial x = X(x-a) - \int (x-a) \partial X$. Ponamus igitur $\partial X = P \partial x$, eritque $\int (x-a) \partial X = \int P(x-a) \partial x$, et ſi iam $P = \frac{\partial X}{\partial x}$, quamdiu x non multum a excedit, vt conſtantem ſpectemus, habebimus $\int P(x-a) \partial x = \frac{1}{2} P(x-a)^2$, ſicque fiet $y = \int X \partial x = b + X(x-a) - \frac{1}{2} P(x-a)^2$, qui valor iam propius ad veritatem accedit, etſi pro X et P ii valores capiantur, quos induunt vel poſito $x = a$, vel poſito $x = a + \alpha$, maiore ſcilicet valore, ad quem hac operatione x creſcere ſtatuimus: ex quo hinc prout vel $x = a$ vel $x = a + \alpha$ ponimus, geminos limites obtinebimus, inter quos veritas ſubſiſtit. Simili autem modo vlterius progredi poterimus:

mus: cum enim P non fit conſtans, erit $\int P(x-a)\partial x = \frac{1}{2}P(x-a)^2 - \frac{1}{2}\int (x-a)^2 \partial P$, vnde ſi ſtatuamus $\partial P = Q\partial x$, erit $\int (x-a)^2 \partial P = \int Q(x-a)^2 \partial x = \frac{1}{3}Q(x-a)^3$, ſi quidem Q vt quantitatem conſtantem ſpectemus, ita vt ſit

$$y = \int X\partial x = b + X(x-a) - \tfrac{1}{2}P(x-a)^2 + \tfrac{1}{2}\cdot\tfrac{1}{3}Q(x-a)^3.$$

Eadem ergo methodo ſi vlterius procedamus, ponendo

$$X = \frac{\partial y}{\partial x};\ P = \frac{\partial X}{\partial x};\ Q = \frac{\partial P}{\partial x};\ R = \frac{\partial Q}{\partial x};\ S = \frac{\partial R}{\partial x};\ \text{etc.}$$

inueniemus

$$y = b + X(x-a) - \tfrac{1}{2}P(x-a)^2 + \tfrac{1}{2.3}Q(x-a)^3 - \tfrac{1}{2.3.4}R(x-a)^4 + \tfrac{1}{2.3.4.5}S(x-a)^5 - \text{etc.}$$

quae ſeries vehementer conuergit, ſi modo x non multum ſuperet a, atque adeo ſi in infinitum continuetur, verum valorem ipſius y exhibebit, ſiquidem in functionibus X, P, Q, R, etc. valor extremus $x = a + \alpha$ ſubſtituatur. Niſi autem eam ſeriem in infinitum extendere velimus, praeſtabit per interualla procedere tribuendo ipſi x ſucceſſiue valores a, a', a'', a''', a'''', etc. ac tum pro ſingulis valores litteris X, P, Q, R, S, etc. conuenientes quaeri oportet, qui ſint, vt ſequuntur:

$$\begin{array}{ll}
\text{ſi fuerit} & x = a,\ a',\ a'',\ a''',\ a^{IV},\ a^{V},\ \text{etc.}\\
\text{fiat} & X = A,\ A',\ A'',\ A''',\ A^{IV},\ A^{V},\ \text{etc.}\\
& \frac{\partial X}{\partial x} = P = B,\ B',\ B'',\ B''',\ B^{IV},\ B^{V},\ \text{etc.}\\
& \frac{\partial P}{\partial x} = Q = C,\ C',\ C'',\ C''',\ C^{IV},\ C^{V},\ \text{etc.}\\
& \frac{\partial Q}{\partial x} = R = D,\ D',\ D'',\ D''',\ D^{IV},\ D^{V},\ \text{etc.}\\
& \text{etc.}
\end{array}$$

tum vero ſit

$$y = b,\ b',\ b'',\ b''',\ b^{IV},\ b^{V},\ \text{etc.}$$

quibus conſtitutis erit, vt ex antecedentibus colligere licet:

b'

$$b' = b + A'(a' - a) - \tfrac{1}{2}B'(a' - a)^2 + \tfrac{1}{6}C'(a' - a)^3 - \tfrac{1}{24}D'(a' - a)^4 + \text{etc.}$$

$$b'' = b' + A''(a'' - a') - \tfrac{1}{2}B''(a'' - a')^2 + \tfrac{1}{6}C''(a'' - a')^3 - \tfrac{1}{24}D''(a'' - a')^4 + \text{etc.}$$

$$b''' = b'' + A'''(a''' - a'') - \tfrac{1}{2}B'''(a''' - a'')^2 + \tfrac{1}{6}C'''(a''' - a'')^3 - \tfrac{1}{24}D'''(a''' - a'')^4 + \text{etc.}$$

$$b^{IV} = b''' + A^{IV}(a^{IV} - a''') - \tfrac{1}{2}B^{IV}(a^{IV} - a''')^2 + \tfrac{1}{6}C^{IV}(a^{IV} - a''')^3 - \tfrac{1}{24}D^{IV}(a^{IV} - a''')^4 + \text{etc.}$$

etc.

quae expreſſiones eousque continuentur, donec pro valore ipſius x quantumuis ab initiali a diſcrepante, valor ipſius y obtineatur.

Corollarium 1.

318. Haec igitur approximandi methodus eo vtitur Theoremate, cuius veritas iam in calculo differentiali eſt demonſtrata, quod ſi y eiusmodi fuerit functio ipſius x, quae poſito $x = a$, fiat $= b$, ac ſtatuatur

$$\frac{\partial y}{\partial x} = X, \quad \frac{\partial X}{\partial x} = P, \quad \frac{\partial P}{\partial x} = Q, \quad \frac{\partial Q}{\partial x} = R, \text{ etc.}$$

fore generaliter:

$$y = b + (x - a) - \tfrac{1}{2}P(x - a)^2 + \tfrac{1}{6}Q(x - a)^3 - \tfrac{1}{24}R(x - a)^4 + \tfrac{1}{120}S(x - a)^5 - \text{etc.}$$

Corollarium 2.

319. Si hanc ſeriem in infinitum continuare vellemus, non opus eſſet, valorem ipſius x parum tantum ab a diuerſum aſſumere. Verum quo iſta ſeries magis conuergens reddatur, expedit ſaltum ab a ad x in interulla diſpeſci, et pro ſingulis operationem hic deſcriptam inſtitui.

Corol-

Corollarium 3.

320. Si valores ipsius x ab a per differentias constantes $= \alpha$ crescere faciamus, sitque vltimus $a + n\alpha = x$, ita vt

$$
\begin{array}{llllll}
\text{si fuerit } x = a, & a+\alpha, & a+2\alpha, & a+3\alpha, & \ldots\ldots & x \\
\text{fiat } X = A, & A', & A'', & A''', & \ldots\ldots & X \\
\frac{\partial X}{\partial x} = P = B, & B', & B'', & B''', & \ldots\ldots & P \\
\frac{\partial P}{\partial x} = Q = C, & C', & C'', & C''', & \ldots\ldots & Q \\
\frac{\partial Q}{\partial x} = R = D, & D', & D'', & D''', & \ldots\ldots & R \\
 & & \text{etc.} & & &
\end{array}
$$

$$
\begin{array}{llllll}
\text{indeque } y = b, & b', & b'', & b''', & \ldots\ldots & y,
\end{array}
$$

erit pro valore $x = x$ omnes series colligendo:

$$
\begin{aligned}
y = b &+ \alpha\,(A' + A'' + A''' + \ldots\ldots + X) \\
&- \tfrac{1}{2}\alpha^2 (B' + B'' + B''' + \ldots\ldots + P) \\
&+ \tfrac{1}{6}\alpha^3 (C' + C'' + C''' + \ldots\ldots + Q) \\
&- \tfrac{1}{24}\alpha^4 (D' + D'' + D''' + \ldots\ldots + R). \\
&\quad\text{etc.}
\end{aligned}
$$

Scholion 1.

321. Demonstratio theorematis Corollario 1. memorati, cui haec methodus approximandi innititur, ex natura differentialium ita instruitur. Sit y functio ipsius x, quae posito $x = a$, fiat $y = b$; et quaeramus valorem ipsius y, si x vtcunque excedat a: incipiamus a valore ipsius maximo, qui est x, et per differentialia descendamus; atque ex differentialibus patet:

ſi fuerit x	fore y
$x - \partial x$	$y - \partial y + \partial\partial y - \partial^3 y + \partial^4 y - \text{etc.}$
$x - 2\partial x$	$y - 2\partial y + 3\partial\partial y - 4\partial^3 y + 5\partial^4 y - \text{etc.}$
$x - 3\partial x$	$y - 3\partial y + 6\partial\partial y - 10\partial^3 y + 15\partial^4 y - \text{etc.}$
.	.
.	.
.	.
.	.
$x - n\partial x$	$y - n\partial y + \frac{n(n+1)}{1 \cdot 2}\partial\partial y - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}\partial^3 y + \frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4}\partial^4 y - \text{etc.}$

Ponamus nunc $x - n\partial x = a$, erit $n = \frac{x-a}{\partial x}$, ideoque numerus infinitus; tum vero valor pro y reſultans per hypotheſin eſſe debet $= b$, quamobrem habebimus

$$b = y - \frac{(x-a)\partial y}{\partial x} + \frac{(x-a)^2 \partial\partial y}{1 \cdot 2\, \partial x^2} - \frac{(x-a)^3 \partial^3 y}{1 \cdot 2 \cdot 3\, \partial x^3} + \frac{(x-a)^4 \partial^4 y}{1 \cdot 2 \cdot 3 \cdot 4\, \partial x^4} - \text{etc.}$$

Quod ſi iam ſtatuamus

$$\frac{\partial y}{\partial x} = X, \quad \frac{\partial X}{\partial x} = P, \quad \frac{\partial P}{\partial x} = Q, \quad \frac{\partial Q}{\partial x} = R, \text{ etc.}$$

reperimus vt ante:

$$y = b + X(x-a) - \tfrac{1}{2}P(x-a)^2 + \tfrac{1}{6}Q(x-a)^3 - \tfrac{1}{24}R(x-a)^4 + \text{etc.}$$

Vnde patet, ſi x quam minime ſuperet a, ſufficere ſtatui $y = b + X(x-a)$, quod eſt fundamentum approximationis primum propoſitae, illius ſcilicet limitis, quo X ex valore maiore ipſius x definitur.

Scholion 2.

322. Quemadmodum hoc ratiocinium nobis alterum tantum limitem ſupra aſſignatum patefecit, ita ad alterum limitem hoc ratiocinium nos manuducet. Scilicet, vti ante ab x ad a deſcendimus, ita nunc ab a ad x aſcendamus

ſi

si abeat a	tum b abibit in
in $a + \partial a$	$b + \partial b$
$a + 2\partial a$	$b + 2\partial b + \partial\partial b$
$a + 3\partial a$	$b + 3\partial b + 3\partial\partial b + \partial^3 b$
.	.
.	.
.	.
$a + n\partial a$	$b + n\partial b + \frac{n(n-1)}{1.2}\partial\partial b + \frac{n(n-1)(n-2)}{1.2.3}\partial^3 b$ etc.

Sit iam $a + n\partial a = x$, seu $n = \frac{x-a}{\partial a}$, et valor ipsius b fiet $= y$. Sint autem A, B, C, D, etc. valores superiorum functionum X, P, Q, R, etc. si loco x scribatur a, eritque pro praesenti casu $A = \frac{\partial b}{\partial a}$; $B = \frac{\partial\partial b}{\partial a^2}$; $C = \frac{\partial^3 b}{\partial a^3}$; etc. Quocirca habebimus

$$y = b + A(x-a) + \tfrac{1}{2}B(x-a)^2 + \tfrac{1}{6}C(x-a)^3 + \tfrac{1}{24}D(x-a)^4 + \text{etc.}$$

quae series superiori praeter signa omnino est similis; ac si x parum excedat a, vt $b + A(x-a)$ satis exacte valorem ipsius y indicet, hinc alter limes supra assignatus nascitur. Quodsi autem progressum ab a ad x, vt supra §. 320. in interualla aequalia secundum differentiam α dispescamus, et termini in singulis seriebus vltimos praecedentes notentur per $'X$, $'P$, $'Q$, $'R$, etc. habebimus pro y quasi alterum limitem

$$\begin{aligned} y = b &+ \alpha\,(A + A' + A'' + \dots + {}'X) \\ &+ \tfrac{1}{2}\alpha^2\,(B + B' + B'' + \dots + {}'P) \\ &+ \tfrac{1}{6}\alpha^3\,(C + C' + C'' + \dots + {}'Q) \\ &+ \tfrac{1}{24}\alpha^4\,(D + D' + D'' + \dots + {}'R) \\ &\text{etc.} \end{aligned}$$

ita vt etiam in hac methodo emendata binos habebimus limites, inter quos verus valor ipsius y contineatur. Propius ergo valorem assequemur, si inter hos limites medium arithme-

 ticum

ticum capiamus ; vnde prodibit

$$y = b + \alpha(A + A' + A'' + \ldots + X) - \tfrac{1}{2}\alpha(A + X) + \tfrac{1}{4}\alpha^2(B - P)$$
$$+ \tfrac{1}{6}\alpha^3(C + C' + C'' + \ldots + Q) - \tfrac{1}{12}\alpha^3(C + Q) + \tfrac{1}{48}\alpha^4(D - R)$$
$$+ \tfrac{1}{120}\alpha^5(E + E' + E'' + \ldots + S) - \tfrac{1}{240}\alpha^5(E + S) + \tfrac{1}{1440}\alpha^6(F - T)$$

etc.

Atque hinc ſuperiores approximationes tantum addendo membrum $\frac{1}{4}\alpha^2(B - P)$, non mediocriter corrigentur.

Exemplum 1.

323. *Logarithmum cuiusuis numeri* x *proxime exprimere.*

Hic igitur eſt $y = \int\frac{\partial x}{x}$, quod integrale ita capitur, vt euaneſcat poſito $x = 1$: erit ergo $a = 1$, $b = 0$ et $X = \frac{1}{x}$. Sumamus iam, ab vnitate ad x per interualla $= \alpha$ aſcendi, et cum ſit $P = \frac{\partial X}{\partial x} = -\frac{1}{xx}$; $Q = \frac{\partial P}{\partial x} = \frac{2}{x^3}$; $R = \frac{\partial Q}{\partial x} = -\frac{6}{x^4}$; pro indicibus

$$\begin{array}{lllllll}
x = & 1; & 1+\alpha; & 1+2\alpha; & 1+3\alpha; & \ldots & x, \text{erit} \\
X = & 1; & \frac{1}{1+\alpha}; & \frac{1}{1+2\alpha}; & \frac{1}{1+3\alpha}; & \ldots & \frac{1}{x} \\
P = & -1; & \frac{1}{(1+\alpha)^2}; & \frac{1}{(1+2\alpha)^2}; & \frac{1}{(1+3\alpha)^2}; & \ldots & -\frac{1}{xx} \\
Q = & 2; & \frac{2}{(1+\alpha)^3}; & \frac{2}{(1+2\alpha)^3}; & \frac{2}{(1+3\alpha)^3}; & \ldots & +\frac{2}{x^3} \\
R = & -6; & \frac{6}{(1+\alpha)^4}; & \frac{6}{(1+2\alpha)^4}; & \frac{6}{(1+3\alpha)^4}; & \ldots & -\frac{6}{x^4}
\end{array}$$

etc.

vnde adipiſcimur

$$lx = \alpha\left[1 + \frac{1}{1+\alpha} + \frac{1}{1+2\alpha} + \frac{1}{1+3\alpha} + \ldots + \frac{1}{x}\right] - \tfrac{1}{2}\alpha\left(1 + \frac{1}{x}\right) - \tfrac{1}{4}\alpha\alpha\left(1 - \frac{1}{xx}\right)$$
$$+ \tfrac{1}{3}\alpha^3\left[1 + \frac{1}{(1+\alpha)^3} + \frac{1}{(1+2\alpha)^3} + \frac{1}{(1+3\alpha)^3} + \ldots + \frac{1}{x^3}\right] - \tfrac{1}{6}\alpha^3\left(1 + \frac{1}{x^3}\right) - \tfrac{1}{8}\alpha^4\left(1 - \frac{1}{x^4}\right)$$
$$+ \tfrac{1}{5}\alpha^5\left[1 + \frac{1}{(1+\alpha)^5} + \frac{1}{(1+2\alpha)^5} + \frac{1}{(1+3\alpha)^5} + \ldots + \frac{1}{x^5}\right] - \tfrac{1}{10}\alpha^5\left(1 + \frac{1}{x^5}\right) - \tfrac{1}{12}\alpha^6\left(1 - \frac{1}{x^6}\right)$$

etc.

Quare ſi ſumamus $\alpha = \frac{1}{m}$, erit

lx

$$l\,x = \tfrac{1}{m} + \tfrac{1}{m+1} + \tfrac{1}{m+2} + \ldots + \tfrac{1}{mx} - \tfrac{(x+1)}{2mx} - \tfrac{(xx-1)}{4mmxx}$$
$$+ \tfrac{1}{3}\left[\tfrac{1}{m^3} + \tfrac{1}{(m+1)^3} + \tfrac{1}{(m+2)^3} + \ldots + \tfrac{1}{(mx)^3}\right] - \tfrac{(x^3+1)}{6m^3x^3} - \tfrac{(x^4-1)}{8m^4x^4}$$
$$+ \tfrac{1}{5}\left[\tfrac{1}{m^5} + \tfrac{1}{(m+1)^5} + \tfrac{1}{(m+2)^5} + \ldots + \tfrac{1}{(mx)^5}\right] - \tfrac{(x^5+1)}{10m^5x^5} - \tfrac{(x^6-1)}{12m^6x^6}$$
etc.

Corollarium.

324. Si hae progreſſiones in infinitum continuentur, erit poſtremarum partium ſumma

$$= -\tfrac{1}{2}\,l\,\tfrac{m}{m-1} - \tfrac{1}{2}\,l\,\tfrac{mx+1}{mx} = -\tfrac{1}{2}\,l\,\tfrac{mx+1}{(m-1)x},$$

primarum vero $= \tfrac{1}{2}\,l\,\tfrac{m+1}{m-1}$: vnde cum ſit

$$l\,x + \tfrac{1}{2}\,l\,\tfrac{mx+1}{(m-1)x} + \tfrac{1}{2}\,l\,\tfrac{m-1}{m+1} = \tfrac{1}{2}\,l\,\tfrac{x(mx+1)}{m+1},$$

erit

$$l\,\tfrac{x(mx+1)}{m+1} = 2\left(\tfrac{1}{m+1} + \tfrac{1}{m+2} + \tfrac{1}{m+3} + \ldots + \tfrac{1}{mx}\right)$$
$$+ \tfrac{2}{3}\left(\tfrac{1}{(m+1)^3} + \tfrac{1}{(m+2)^3} + \tfrac{1}{(m+3)^3} + \ldots + \tfrac{1}{m^3x^3}\right)$$
$$+ \tfrac{2}{5}\left(\tfrac{1}{(m+1)^5} + \tfrac{1}{(m+2)^5} + \tfrac{1}{(m-3)^5} + \ldots + \tfrac{1}{m^5x^5}\right)$$
etc.

quae expreſſio adeo, ſi in infinitum continuetur, verum valorem log. $\frac{x(mx+1)}{m+1}$ praebet.

Exemplum 2.

325. *Arcum circuli cuius tangens eſt* $= \frac{x}{c}$ *hac methodo proxime exprimere.*

Quaeſtio igitur eſt de integrali $y = \int \frac{c\,\partial x}{cc+xx}$, quod poſito $x = 0$ euaneſcit; eritque $a = 0$, et $b = 0$, tum vero

$$X = \tfrac{c}{cc+xx};\quad P = \tfrac{\partial X}{\partial x} = \tfrac{-2cx}{(cc+xx)^2};\quad Q = \tfrac{\partial P}{\partial x} = \tfrac{-2c(cc-3xx)}{(cc+xx)^3};$$
$$R = \tfrac{\partial Q}{\partial x} = \tfrac{6cx(3cc-4xx)}{(cc+xx)^4};\quad S = \tfrac{\partial R}{\partial x} = \tfrac{6c(3c^4-33ccxx+20x^4)}{(cc+xx)^5};\ \text{etc.}$$

quae formae in infinitum continuatae dant

$$y = \frac{cx}{cc+xx} + \frac{cx^3}{(cc+xx)^2} - \frac{cx^3(cc-3xx)}{3(cc+xx)^3} - \frac{cx^5(3cc-4xx)}{4(cc+xx)^4}$$
$$+ \frac{cx^5(3c^4-33ccxx+20x^4)}{20(cc+xx)^5} + \text{etc.}$$

Verum ſi x per interualla $= 1$, vt ſit $\alpha = 1$, creſcere ponamus, erit

$$\mathrm{A} = \frac{c}{cc}; \quad \mathrm{B} = 0; \quad \mathrm{C} = \frac{-2c^3}{c^6}; \quad \mathrm{D} = 0;$$
$$\mathrm{A}' = \frac{c}{cc+1}; \quad \mathrm{B}' = \frac{-2c}{(cc+1)^2}; \quad \mathrm{C}' = \frac{-2c(cc-3)}{(cc+1)^3}; \quad \mathrm{D}' = \frac{6c(3cc-4)}{(cc+1)^4};$$
$$\mathrm{A}'' = \frac{c}{cc+4}; \quad \mathrm{B}'' = \frac{-4c}{(cc+4)^2}; \quad \mathrm{C}'' = \frac{-2c(cc-12)}{(cc+4)^3}; \quad \mathrm{D}'' = \frac{12c(3cc-16)}{(cc+4)^4};$$
$$\mathrm{A}''' = \frac{c}{cc+9}; \quad \mathrm{B}''' = \frac{-6c}{(cc+9)^2}; \quad \mathrm{C}''' = \frac{-2c(cc-27)}{(cc+9)^3}; \quad \mathrm{D}''' = \frac{18c(3cc-36)}{(cc+9)^4};$$
$$\vdots$$
$$\mathrm{X} = \frac{c}{cc+xx}; \quad \mathrm{P} = \frac{-2cx}{(cc+xx)^2}; \quad \mathrm{Q} = \frac{-2c(cc-3xx)}{(cc+xx)^3}; \quad \mathrm{R} = \frac{6cx(3cc-4xx)}{(cc+xx)^4};$$

hincque

$$y = c\left(\frac{1}{cc} + \frac{1}{cc+1} + \frac{1}{cc+4} + \frac{1}{cc+9} + \ldots + \frac{1}{cc+xx}\right) - \frac{1}{2c} - \frac{c}{2(cc+xx)} + \frac{cx}{2(cc+xx)^2}$$
$$-\frac{c}{3}\left(\frac{1}{c^4} + \frac{cc-3}{(cc+1)^3} + \frac{cc-12}{(cc+4)^3} + \frac{cc-27}{(cc+9)^3} + \ldots + \frac{cc-3xx}{(cc+xx)^3}\right) + \frac{1}{6c^3} + \frac{c(cc-3xx)}{6(cc+xx)^3} - \frac{cx(3cc-4xx)}{8(cc+xx)^4}$$
$$\text{etc.}$$

Corollarium.

326. Poſito ergo $c = x = 4$, vt fiat $y = \text{Ang. tang. } 1 = \frac{\pi}{4}$, erit

$$\frac{\pi}{4} = \frac{1}{4} + \frac{4}{17} + \frac{4}{20} + \frac{4}{25} + \frac{1}{8} - \frac{1}{8} - \frac{1}{16} + \frac{1}{128}$$
$$-\frac{4}{3}\left(\frac{1}{256} + \frac{13}{17^3} + \frac{4}{20^3} - \frac{11}{25^3} - \frac{32}{32^3}\right) + \frac{1}{384} - \frac{1}{1536} + \frac{1}{32768},$$

cuius valor non multum a veritate diſcedit; ſed haec exempla tantum

tantum illustrationis causa afferro, non vt approximatio facilior, quam aliae methodi suppeditant, inde expectetur.

Exemplum 3.

327. *Integrale* $y = \int \frac{e^{-\frac{1}{x}} \partial x}{x}$ *ita sumtum, vt euanescet posito* $x = 0$, *vero proxime assignare.*

Per reductiones supra expositas est

$$\int \frac{e^{-\frac{1}{x}} \partial x}{x} = e^{-\frac{1}{x}} x - \int e^{-\frac{1}{x}} \partial x,$$

et pars $e^{-\frac{1}{x}} x$ euanescit, posito $x = 0$. Quaeramus ergo integrale $z = \int e^{-\frac{1}{x}} \partial x$, quia eo inuento habetur $y = e^{-\frac{1}{x}} x - z$; ac supra iam obseruauimus, alias methodos approximandi in hoc exemplo frustra tentari. Cum igitur, posito $x = 0$, euanescat z, erit $a = 0$ et $b = 0$, tum vero $X = e^{-\frac{1}{x}}$, hincque $P = \frac{\partial X}{\partial x} = e^{-\frac{1}{x}} \frac{1}{xx}$; $Q = \frac{\partial P}{\partial x} = e^{-\frac{1}{x}} (\frac{1}{x^4} - \frac{2}{x^3})$; $R = \frac{\partial Q}{\partial x} = e^{-\frac{1}{x}} (\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4})$; $S = \frac{\partial R}{\partial x} = e^{-\frac{1}{x}} (\frac{1}{x^8} - \frac{12}{x^7} + \frac{36}{x^6} - \frac{24}{x^5})$ etc. quibus valoribus in infinitum continuatis, erit

$$\left. \begin{array}{l} z = e^{-\frac{1}{x}} (x - \frac{1}{2} + \frac{1}{6} x^3 (\frac{1}{x^4} - \frac{2}{x^3}) - \frac{1}{24} x^4 (\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}) \\ \qquad + \frac{1}{120} x^5 (\frac{1}{x^8} - \frac{12}{x^7} + \frac{36}{x^6} - \frac{24}{x^5}) \end{array} \right\} \text{seu}$$

$$\left. \begin{array}{l} z = e^{-\frac{1}{x}} (x - \frac{1}{2} + \frac{1}{6}(\frac{1}{x} - 2) - \frac{1}{24}(\frac{1}{xx} - \frac{6}{x} + 6) + \frac{1}{120}(\frac{1}{x^3} - \frac{12}{x^2} + \frac{36}{x} - 24) \\ \qquad - \frac{1}{720} (\frac{1}{x^4} - \frac{20}{x^3} + \frac{120}{x^2} - \frac{240}{x} + 120) \text{ etc.} \end{array} \right\},$$

quae series parum conuergit, quicunque valor ipsi x tribuatur. Per interualla igitur a 0 vsque ad x ascendamus, ponendo

nendo pro x ſucceſſiue 0, α, 2α, 3α, etc. vbi notandum fore $A = 0$, $B = 0$, $C = 0$, $D = 0$, etc. ac regula noſtra praebet:

$$z = \alpha\left(e^{-\frac{1}{\alpha}} + e^{-\frac{1}{2\alpha}} + e^{-\frac{1}{3\alpha}} + \ldots + e^{-\frac{1}{x}}\right) - \tfrac{1}{2}\alpha e^{-\frac{1}{x}} - \tfrac{1}{4}\alpha^2 e^{-\frac{1}{x}} \tfrac{1}{xx}$$
$$+ \tfrac{1}{6}\alpha^3 \left[e^{-\frac{1}{\alpha}}\left(\tfrac{1}{\alpha^4} - \tfrac{2}{\alpha^3}\right) + e^{-\frac{1}{2\alpha}}\left(\tfrac{1}{16\alpha^4} - \tfrac{2}{8\alpha^3}\right) + e^{-\frac{1}{3\alpha}}\left(\tfrac{1}{81\alpha^4} - \tfrac{2}{27\alpha^3}\right) \ldots\right.$$
$$\left. + e^{-\frac{1}{x}}\left(\tfrac{1}{x^4} - \tfrac{2}{x^3}\right)\right] - \tfrac{1}{12}\alpha^3 e^{-\frac{1}{x}}\left(\tfrac{1}{x^4} - \tfrac{2}{x^3}\right) - \tfrac{1}{48}\alpha^4 e^{-\frac{1}{x}}\left(\tfrac{1}{x^6} - \tfrac{6}{x^5} + \tfrac{6}{x^4}\right).$$

Si hinc valorem ipſius z pro caſu $x = 1$ determinare velimus, et pro α fractionem paruam $\frac{1}{n}$ aſſumamus, habebimus:

$$z = \tfrac{1}{n}\left(e^{-\frac{n}{1}} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + e^{-\frac{n}{4}} + \ldots + e^{-\frac{n}{n}}\right) - \tfrac{1}{2ne} - \tfrac{1}{4nne}$$
$$+ \tfrac{1}{6}\left[e^{-\frac{n}{1}}\left(\tfrac{n-2}{1}\right) + e^{-\frac{n}{2}}\left(\tfrac{n-4}{16}\right) + e^{-\frac{n}{3}}\left(\tfrac{n-6}{81}\right) + \ldots + e^{-\frac{n}{n}}\left(\tfrac{n-2n}{n^4}\right)\right]$$
$$+ \tfrac{1}{12 n^3 e} - \tfrac{1}{48 n^4 e}.$$

Si hic pro n ſumatur numerus mediocriter magnus vel vti 10, valor ipſius z ad partem millioneſimam vnitatis exactus reperitur, ac vicies exactior prodiret, ſi pro n ſumeremus 20.

Scholion 1.

328. Hoc exemplum ſufficiat eximium vſum huius methodi approximandi oſtendiſſe. Interim tamen occurrunt caſus, quibus ne hac quidem methodo vti licet, etiamſi totum ſpatium, per quod variabilis x creſcit, in minima interualla diuidamus. Euenit hoc, quando functio X pro quopiam interuallo, dum variabili x certus quidam valor tribuitur, in infinitum excreſcit, cum tamen ipſa quantitas integralis $y = \int X \partial x$ hoc caſu non fiat infinita: veluti ſi fuerit $y = \int \frac{\partial x}{\sqrt{(a-x)}}$, vbi $X = \frac{1}{\sqrt{(a-x)}}$, quae poſito $x = a$ fit infinita, integrale vero $y = C - 2\sqrt{(a-x)}$ hoc caſu eſt finitum.

Hoc

Hoc autem semper vsu venit, quoties huiusmodi factor $a-x$ in denominatore habet exponentem vnitate minorem, tum enim idem factor in integrali in numeratorem transit; sin autem eiusdem factoris exponens in denominatore est vnitas, vel adeo vnitate maior, tum etiam ipsum integrale casu $x=a$ fit infinitum, quo casu quia approximatio cessat, hic tantum de iis sermo est, vbi exponens vnitate est minor; quoniam tum approximatio reuera turbatur. Verum huic incommodo facile medela afferri potest, cum enim differentiale eiusmodi formam sit habiturum $\frac{X\partial x}{(a-x)^{\lambda:\mu}}$, existente $\lambda<\mu$, ponatur $a-x=z^\mu$, vt sit $x=a-z^\mu$ et $\partial x=-\mu z^{\mu-1}\partial z$, et differentiale nostrum erit $=-\mu X z^{\mu-\lambda-1}\partial z$, quod casu $x=a$ seu $z=0$, non amplius fit infinitum. Vel quod eodem redit, pro iis interuallis, quibus functio X fit infinita, integratio seorsim reuera instituatur, ponendo $x=a\pm\omega$, tum enim formula $X\partial x$ satis fiet simplex ob ω valde paruum, vt integratio nihil habeat difficultatis. Veluti si valorem ipsius $y=\int\frac{xx\partial x}{\sqrt{(a^4-x^4)}}$ per interualla ab $x=0$ vsque ad $x=a-\alpha$, iam simus consecuti, pro hoc vltimo interuallo ponamus $x=a-\omega$, et integrari oportebit $\frac{(a-\omega)^2\partial\omega}{\sqrt{(4a^3\omega-6aa\omega\omega+4a\omega^3-\omega^4)}}$, quod ob ω valde paruum abit in

$$\frac{\partial\omega\sqrt{a}}{2\sqrt{\omega}}\left(1-\frac{\omega}{2a}+\frac{7\omega\omega}{8aa}\right),$$

cuius integrale, sumto $\omega=\alpha$, est

$$\sqrt{a\alpha}-\frac{\alpha\sqrt{\alpha}}{6\sqrt{a}}+\frac{7\alpha\alpha\sqrt{\alpha}}{4aa\sqrt{a}},$$

quod si ad plures terminos continuetur, non solum pro vltimo interuallo sed pro duobus pluribusue postremis, ponendo $\omega=2\alpha$ vel $\omega=3\alpha$ adhiberi potest. Pro quibus enim interuallis denominator iam fit satis paruus, praestat hac methodo vti, quam ea quae ante est exposita.

Scholion 2.

329. Interdum etiam illud incommodum occurrit, vt denominator duobus casibus euanescat, veluti si fuerit $y = \int \frac{X \partial x}{\sqrt{(a-x)(x-b)}}$, vbi variabilis x semper inter limites b et a contineri debet, ita vt cum a b ad a creuerit, deinceps iterum ab a ad b decrescat; interea autem integrale y continuo crescere pergat, cuius igitur valor per interualla commode determinari non potest. Hoc ergo casu in subsidium vocetur haec substitutio $x = \frac{1}{2}(a+b) - \frac{1}{2}(a-b)\,\text{cos.}\,\Phi$, qua fit $\partial x = + \frac{1}{2}(a-b)\,\partial\Phi\,\text{sin.}\,\Phi$, et $(a-x)(x-b) = [\frac{1}{2}(a-b) + \frac{1}{2}(a-b)\,\text{cos.}\,\Phi][\frac{1}{2}(a-b) - \frac{1}{2}(a-b)\,\text{cos.}\,\Phi]$, seu $(a-x)(x-b) = \frac{1}{4}(a-b)^2\,\text{sin.}\,\Phi^2$: vnde oritur $y = \int X\,\partial\Phi$, quae nullo amplius incommodo laborat, cum angulum Φ continuo vlterius aequabiliter augere licet. Hoc etiam ad casus patet, vbi bini factores in denominatore non eundem habent exponentem, veluti si fuerit $y = \int \frac{X \partial x}{\sqrt[2\lambda]{(a-x)^\mu (x-b)^\nu}}$, ita vt μ et ν sint minores quam 2λ, quem exponentem parem suppono. Si iam μ et ν non sint aequales sed $\nu < \mu$, ad aequalitatem reducantur hoc modo, $y = \int \frac{X \partial x \sqrt[2\lambda]{(x-b)^{\mu-\nu}}}{\sqrt[2\lambda]{(a-x)^\mu (x-b)^\mu}}$. Quodsi iam vt ante ponatur $x = \frac{1}{2}(a+b) - \frac{1}{2}(a-b)\,\text{cos.}\,\Phi$, obtinebitur

$$y = \left(\frac{a-b}{2}\right)^{\frac{2\lambda-\mu-\nu}{2\lambda}} \int X\,\partial\Phi\,\text{sin.}\,\Phi^{\frac{\lambda-\mu}{\lambda}} (1 - \text{cof.}\,\Phi)^{\frac{\mu-\nu}{2\lambda}},$$

vbi angulum Φ quousque libuerit continuare et methodo per interualla procedente vti licet. Quibus obseruatis vix quicquam amplius hanc methodum approximandi remorabitur.

CA-

CAPVT VIII.

DE VALORIBVS INTEGRALIVM QVOS CERTIS TANTVM CASIBVS RECIPIVNT.

Problema 38.

330.

Integralis $\int \frac{x^m \partial x}{\sqrt{(1-xx)}}$ valorem, quem pofito $x = 1$ recipit, aſſignare, integrali ſcilicet ita determinato, vt euaneſcat poſito $x = 0$.

Solutio.

Pro caſibus ſimpliciſſimis, quibus $m = 0$ vel $m = 1$, habemus poſito $x = 1$, poſt integrationem

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2} \text{ et } \int \frac{x \partial x}{\sqrt{(1-xx)}} = 1.$$

Deinde ſupra §. 119. vidimus eſſe in genere

$$\int \frac{x^{m+1} \partial x}{\sqrt{(1-xx)}} = \frac{m}{m+1} \int \frac{x^{m-1} \partial x}{\sqrt{(1-xx)}} - \frac{1}{m+1} x^m \sqrt{(1-xx)};$$

caſu ergo $x = 1$ erit

$$\int \frac{x^{m+1} \partial x}{\sqrt{(1-xx)}} = \frac{m}{m+1} \int \frac{x^{m-1} \partial x}{\sqrt{(1-xx)}},$$

vnde a ſimpliciſſimis ad maiores exponentis m valores progrediendo obtinebimus:

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2}$$

$$\int \frac{x^2 \partial x}{\sqrt{(1-xx)}} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int \frac{x^4 \partial x}{\sqrt{(1-xx)}} = \frac{1.3}{2.4} \cdot \frac{\pi}{2}$$

$$\int \frac{x^6 \partial x}{\sqrt{(1-xx)}} = \frac{1.3.5}{2.4.6} \cdot \frac{\pi}{2}$$

$$\int \frac{x^8 \partial x}{\sqrt{(1-xx)}} = \frac{1.3.5.7}{2.4.6.8} \cdot \frac{\pi}{2}$$

$$\vdots$$

$$\int \frac{x^{2n} \partial x}{\sqrt{(1-xx)}} = \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots \quad 2n} \cdot \frac{\pi}{2}$$

$$\int \frac{x \partial x}{\sqrt{(1-xx)}} = 1$$

$$\int \frac{x^3 \partial x}{\sqrt{(1-xx)}} = \frac{2}{3}$$

$$\int \frac{x^5 \partial x}{\sqrt{(1-xx)}} = \frac{2.4}{3.5}$$

$$\int \frac{x^7 \partial x}{\sqrt{(1-xx)}} = \frac{2.4.6}{3.5.7}$$

$$\int \frac{x^9 \partial x}{\sqrt{(1-xx)}} = \frac{2.4.6.8}{3.5.7.9}$$

$$\vdots$$

$$\int \frac{x^{2n+1} \partial x}{\sqrt{(1-xx)}} = \frac{2.4.6 \ldots \quad 2n}{3.5.7 \ldots (2n+1)}.$$

Corollarium 1.

331. Integrale ergo $\int \frac{x^m \partial x}{\sqrt{(1-xx)}}$, posito $x=1$, algebraice exprimitur casibus, quibus exponens m est numerus integer impar; casibus autem quibus est par quadraturam circuli inuoluit; semper enim π designat peripheriam circuli, cuius diameter $=1$.

Corollarium 2.

332. Si binas postremas formulas in se multiplicemus prodit:

$\int x$

$$\int \frac{x^{2n}\,\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^{2n+1}\,\partial x}{\sqrt{(1-xx)}} = \frac{1}{2n+1} \cdot \frac{\pi}{2},$$

posito scilicet $x = 1$, quam veram esse patet, etiamsi n non sit numerus integer.

Corollarium 3.

333. Haec ergo aequalitas subsistet, si ponamus $x = z^{\nu}$, iisdem conditionibus, quia sumto $x = 0$ vel $x = 1$ fit $z = 0$ vel $z = 1$. Erit ergo

$$\nu\nu \int \frac{z^{2n\nu+\nu-1}\,\partial z}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{2n\nu+2\nu-1}\,\partial z}{\sqrt{(1-z^{2\nu})}} = \frac{1}{2n+1} \cdot \frac{\pi}{2},$$

et posito $2n\nu + \nu - 1 = \mu$, fiet posito $z = 1$

$$\int \frac{z^{\mu}\,\partial z}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{\mu+\nu}\,\partial z}{\sqrt{(1-z^{2\nu})}} = \frac{1}{\nu(\mu+1)} \cdot \frac{\pi}{2}.$$

Scholion 1.

334. Quod tale productum binorum integralium exhiberi queat, eo magis est notatu dignum, quod aequalitas haec subsistit, etiamsi neutra formula neque algebraice neque per π exhiberi queat. Veluti si $\nu = 2$ et $\mu = 0$, fit

$$\int \frac{\partial z}{\sqrt{(1-z^4)}} \cdot \int \frac{zz\,\partial z}{\sqrt{(1-z^4)}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4},$$

similique modo:

$\nu = 3,\ \mu = 0$ fit $\int \frac{\partial z}{\sqrt{(1-z^6)}} \cdot \int \frac{z^3\,\partial z}{\sqrt{(1-z^6)}} = \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6};$

$\nu = 3,\ \mu = 1$ fit $\int \frac{z\,\partial z}{\sqrt{(1-z^6)}} \cdot \int \frac{z^4\,\partial z}{\sqrt{(1-z^6)}} = \frac{1}{6} \cdot \frac{\pi}{2} = \frac{\pi}{12};$

$\nu = 4,\ \mu = 0$ fit $\int \frac{\partial z}{\sqrt{(1-z^8)}} \cdot \int \frac{z^4\,\partial z}{\sqrt{(1-z^8)}} = \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8};$

$\nu = 4,\ \mu = 2$ fit $\int \frac{zz\,\partial z}{\sqrt{(1-z^8)}} \cdot \int \frac{z^6\,\partial z}{\sqrt{(1-z^8)}} = \frac{1}{12} \cdot \frac{\pi}{2} = \frac{\pi}{24};$

$\nu = 5,\ \mu = 0$ fit $\int \frac{\partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^5\,\partial z}{\sqrt{(1-z^{10})}} = \frac{1}{5} \cdot \frac{\pi}{2} = \frac{\pi}{10};$

$\nu = 5,\ \mu = 1$ fit $\int \frac{z\,\partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^6\,\partial z}{\sqrt{(1-z^{10})}} = \frac{1}{10} \cdot \frac{\pi}{2} = \frac{\pi}{20};$

$$\nu = 5,\ \mu = 2 \text{ fit } \int \frac{zz\,\partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^7\,\partial z}{\sqrt{(1-z^{10})}} = \tfrac{1}{15} \cdot \tfrac{\pi}{2} = \tfrac{\pi}{30};$$

$$\nu = 5,\ \mu = 3 \text{ fit } \int \frac{z^3\,\partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^8\,\partial z}{\sqrt{(1-z^{10})}} = \tfrac{1}{20} \cdot \tfrac{\pi}{2} = \tfrac{\pi}{40};$$

quae Theoremata ſine dubio omni attentione ſunt digna.

Scholion 2.

335. Facile hinc etiam colligitur valor integralis $\int \frac{x^m\,\partial x}{\sqrt{(x - xx)}}$ poſito $x = 1$, ſi enim ſcribamus $x = zz$, fiet hoc integrale $2\int \frac{z^{2m}\,\partial z}{\sqrt{(1 - zz)}}$; quocirca pro caſu $x = 1$ nanciſcimur ſequentes valores:

$$\begin{array}{l|l}
\int \frac{\partial x}{\sqrt{(x-xx)}} = \pi & \int \frac{x^4\,\partial x}{\sqrt{(x-xx)}} = \frac{1.3.5.7}{2.4.6.8}\,\pi; \\
\int \frac{x\,\partial x}{\sqrt{(x-xx)}} = \frac{1}{2}\cdot\pi & \int \frac{x^5\,\partial x}{\sqrt{(x-xx)}} = \frac{1.3.5.7.9}{2.4.6.8.10}\,\pi; \\
\int \frac{x^2\,\partial x}{\sqrt{(x-xx)}} = \frac{1.3}{2.4}\cdot\pi & \quad\vdots \\
\int \frac{x^3\,\partial x}{\sqrt{(x-xx)}} = \frac{1.3.5}{2.4.6}\cdot\pi & \int \frac{x^m\,\partial x}{\sqrt{(x-xx)}} = \frac{1.3.5\ldots(2m-1)}{2.4.6\ldots\ 2m}\,\pi.
\end{array}$$

Hinc ergo integralium huiusmodi formulas inuoluentium, quae magis ſunt complicata, valores, quos poſito $x = 1$ recipiunt, per ſeries ſuccincte exprimi poſſunt, quem vſum aliquot exemplis declaremus.

Exemplum 1.

336. *Valorem integralis* $\int \frac{\partial x}{\sqrt{(1-x^4)}}$, *poſito* $x = 1$, *per ſeriem exhibere.*

Integrali detur haec forma $\int \frac{\partial x}{\sqrt{(1-xx)}} \cdot (1 + xx)^{-\frac{1}{2}}$, vt habeamus

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} = \int \frac{\partial x}{\sqrt{(1-xx)}} \left(1 - \tfrac{1}{2} xx + \tfrac{1.3}{2.4} x^4 - \tfrac{1.3.5}{2.4.6} x^6 + \tfrac{1.3.5.7}{2.4.6.8} x^8 - \text{etc.}\right)$$

ſingulis ergo terminis pro caſu $x = 1$ integratis, orietur

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} = \frac{\pi}{2} \left(1 - \tfrac{1}{4} + \tfrac{1.9}{4.16} - \tfrac{1.9.25}{4.16.36} + \tfrac{1.9.25.49}{4.16.36.64} - \text{etc.}\right)$$

Corollarium.

337. Simili modo pro eodem caſu $x = 1$ reperitur:

$$\int \frac{x\partial x}{\sqrt{(1-x^4)}} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.} = \frac{\pi}{4}$$

$$\int \frac{xx\partial x}{\sqrt{(1-x^4)}} = \frac{\pi}{2} \left(\tfrac{1}{2} - \tfrac{1^2.3}{2^2.4} + \tfrac{1^2.3^2.5}{2^2.4^2.6} - \tfrac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8} + \text{etc.}\right)$$

$$\int \frac{x^3\partial x}{\sqrt{(1-x^4)}} = \tfrac{2}{3} - \tfrac{4}{3.5} + \tfrac{6}{5.7} - \tfrac{8}{7.9} + \tfrac{10}{9.11} - \text{etc.}$$

eſt autem $\int \frac{x^3\partial x}{\sqrt{(1-x^4)}} = \frac{1}{2} - \frac{1}{2}\sqrt{(1-x^4)}$, ideoque $= \frac{1}{2}$, poſito $x = 1$, vnde haec poſtrema ſeries $= \frac{1}{2}$, quod manifeſtum eſt.

Exemplum 2.

338. *Valorem integralis* $\int \partial x \sqrt{\frac{1+axx}{1-xx}}$, *caſu* $x = 1$, *per ſeriem exhibere.*

Cum ſit

$$\sqrt{(1+axx)} = 1 + \tfrac{1}{2} axx - \tfrac{1.1}{2.4} a^2 x^4 + \tfrac{1.1.3}{2.4.6} a^3 x^6 - \text{etc.}$$

erit per $\int \frac{\partial x}{\sqrt{(1-xx)}}$ multiplicando et integrando

$$\int \partial x \sqrt{\frac{1+axx}{1-xx}} = \frac{\pi}{2} \left(1 + \tfrac{1.1}{2.2} a - \tfrac{1.1.1.3}{2.2.4.4} a^2 + \tfrac{1.1.1.3.3.5}{2.2.4.4.6.6} a^3 - \text{etc.}\right)$$

vnde peripheriam ellipſis cognoſcere licet.

Exemplum 3.

339. *Valorem integralis* $\int \frac{\partial x}{\sqrt{x(1-xx)}}$, *caſu* $x = 1$, *per ſeriem exhibere.*

Repraeſentetur haec formula ita $\int \frac{\partial x (1+x)^{-\frac{1}{2}}}{\sqrt{(x-xx)}}$, vt ſit $=$

$$\int \frac{\partial x}{\sqrt{(x-xx)}} \left(1 - \tfrac{1}{2} x + \tfrac{1.3}{2.4} x^2 - \tfrac{1.3.5}{2.4.6} x^3 + \text{etc.}\right):$$

vnde

vnde feries haec obtinetur:

$$\int \frac{\partial x}{\sqrt{x(1-xx)}} = \pi\left(1 - \frac{1}{4} + \frac{1.\ 9}{4.\ 16} - \frac{1.\ 9.\ 25}{4.\ 16.\ 36} + \text{etc.}\right)$$

quae ab exemplo primo haud differt: quod non mirum, cum pofito $x = zz$, haec formula ad illam reducatur.

Problema 39.

340. Valorem integralis $\int x^{m-1}\partial x(1-xx)^{n-\frac{1}{2}}$, quod pofito $x=0$ euanefcat, definire cafu $x=1$.

Solutio.

Reductiones fupra §. 118. datae praebent pro hoc cafu

$$\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}+1} = \frac{x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2}$$

$$+ \frac{\mu+2}{m+\mu+2}\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}}:$$

fumto ergo $\mu = 2n-1$, erit

$$\int x^{m-1}\partial x(1-xx)^{n+\frac{1}{2}} = \frac{2n+1}{m+2n+1}\int x^{m-1}\partial x(1-xx)^{n-\frac{1}{2}}$$

pofito $x=1$. Cum igitur in praecedente problemate valor $\int \frac{x^{m-1}\partial x}{\sqrt{(1-xx)}}$ fit affignatus, quem breuitatis gratia ponamus $=\text{M}$, hinc ad fequentes progrediamur:

$$\int \frac{x^{m-1}\partial x}{\sqrt{(1-xx)}} = \text{M};$$

$$\int x^{m-1}\partial x(1-xx)^{\frac{1}{2}} = \frac{1}{m+1}\text{M};$$

$$\int x^{m-1}\partial x(1-xx)^{\frac{3}{2}} = \frac{1.\ 3}{(m+1)(m+3)}\text{M};$$

$$\int x^{m-1}\partial x(1-xx)^{\frac{5}{2}} = \frac{1.\ 3.\ 5}{(m+1)(m+3)(m+5)};$$

et

et in genere

$$\int x^{m-1}\partial x(1-xx)^{n-\frac{1}{2}} = \frac{1. \quad 3. \quad 5. \quad \ldots \quad (2n-1)}{(m+1)(m+3)(m+5)\ldots(m+2n-1)} M.$$

Iam duo caſus ſunt perpendendi, prout $m-1$ eſt vel numerus par vel impar: ſi enim

$m-1$ ſit par, erit $M = \frac{1.3.5.\ldots\ldots(m-2)}{2.4.6.\ldots\ldots(m-1)} \cdot \frac{\pi}{2}$;

$m-1$ ſit impar, erit $M = \frac{2.4.6\ldots\ldots(m-2)}{3.5.7\ldots\ldots(m-1)}$.

Hinc ſequentes deducuntur valores:

$\int \partial x \sqrt{(1-xx)} = \frac{\pi}{4}$	$\int x \partial x \sqrt{(1-xx)} = \frac{1}{3}$
$\int x^2 \partial x \sqrt{(1-xx)} = \frac{1}{4} \cdot \frac{\pi}{4}$	$\int x^3 \partial x \sqrt{(1-xx)} = \frac{1}{3} \cdot \frac{2}{5}$
$\int x^4 \partial x \sqrt{(1-xx)} = \frac{1.3}{4.6} \cdot \frac{\pi}{4}$	$\int x^5 \partial x \sqrt{(1-xx)} = \frac{1}{3} \cdot \frac{2.4}{5.7}$
$\int x^6 \partial x \sqrt{(1-xx)} = \frac{1.3.5}{4.6.8} \cdot \frac{\pi}{4}$	$\int x^7 \partial x \sqrt{(1-xx)} = \frac{1}{3} \cdot \frac{2.4.6}{5.7.9}$

$\int \partial x (1-xx)^{\frac{3}{2}} = \frac{3\pi}{16}$	$\int x \partial x (1-xx)^{\frac{3}{2}} = \frac{1}{5}$
$\int xx \partial x (1-xx)^{\frac{3}{2}} = \frac{1}{6} \cdot \frac{3\pi}{16}$	$\int x^3 \partial x (1-xx)^{\frac{3}{2}} = \frac{1}{5} \cdot \frac{2}{7}$
$\int x^4 \partial x (1-xx)^{\frac{3}{2}} = \frac{1.3}{6.8} \cdot \frac{3\pi}{16}$	$\int x^5 \partial x (1-xx)^{\frac{3}{2}} = \frac{1}{5} \cdot \frac{2.4}{7.9}$
$\int x^6 \partial x (1-xx)^{\frac{3}{2}} = \frac{1.3.5}{6.8.10} \cdot \frac{3\pi}{16}$	$\int x^7 \partial x (1-xx)^{\frac{3}{2}} = \frac{1}{5} \cdot \frac{2.4.6}{7.9.11}$

$\int \partial x (1-xx)^{\frac{5}{2}} = \frac{5\pi}{32}$	$\int x \partial x (1-xx)^{\frac{5}{2}} = \frac{1}{7}$
$\int x^2 \partial x (1-xx)^{\frac{5}{2}} = \frac{1}{8} \cdot \frac{5\pi}{32}$	$\int x^3 \partial x (1-xx)^{\frac{5}{2}} = \frac{1}{7} \cdot \frac{2}{9}$
$\int x^4 \partial x (1-xx)^{\frac{5}{2}} = \frac{1.3}{8.10} \cdot \frac{5\pi}{32}$	$\int x^5 \partial x (1-xx)^{\frac{5}{2}} = \frac{1}{7} \cdot \frac{2.4}{9.11}$
$\int x^6 \partial x (1-xx)^{\frac{5}{2}} = \frac{1.3.5}{8.10.12} \cdot \frac{5\pi}{32}$	$\int x^7 \partial x (1-xx)^{\frac{5}{2}} = \frac{1}{7} \cdot \frac{2.4.6}{9.11.13}$

etc.

Problema 40.

341. Valores integralium $\int \frac{x^m \partial x}{\sqrt[3]{(1-x^3)}}$ et $\int \frac{x^m \partial x}{\sqrt[3]{(1-x^3)^2}}$, posito $x = 1$, assignare.

Solutio.

Ponamus pro casibus simplicissimis:

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = A; \int \frac{x \partial x}{\sqrt[3]{(1-x^3)}} = B; \int \frac{x x \partial x}{\sqrt[3]{(1-x^3)}} = C$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = A'; \int \frac{x \partial x}{\sqrt[3]{(1-x^3)^2}} = B'; \int \frac{x x \partial x}{\sqrt[3]{(1-x^3)^2}} = C'$$

et ex reductione prima §. 118. posito $a = 1$ et $b = -1$, pro casu $x = 1$ habemus

$$\int x^{m+n-1} \partial x (1-x^n)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu + n\mu + n\nu} \int x^{m-1} \partial x (1-x^n)^{\frac{\mu}{\nu}},$$

ergo pro priori vbi $n = 3$, $\nu = 3$ et $\mu = -1$,

$$\int x^{m+2} \partial x (1-x^3)^{-\frac{1}{3}} = \frac{m}{m+2} \int x^{m-1} \partial x (1-x^3)^{-\frac{1}{3}}$$

et pro posteriori, vbi $n = 3$, $\nu = 3$ et $\mu = -2$

$$\int x^{m+2} \partial x (1-x^3)^{-\frac{2}{3}} = \frac{m}{m+1} \int x^{m-1} \partial x (1-x^3)^{-\frac{2}{3}},$$

hinc obtinemus pro forma priori:

$\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = A$	$\int \frac{x \partial x}{\sqrt[3]{(1-x^3)}} = B$	$\int \frac{x x \partial x}{\sqrt[3]{(1-x^3)}} = C$
$\int \frac{x^3 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1}{3} A$	$\int \frac{x^4 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2}{4} B$	$\int \frac{x^5 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3}{3} C$
$\int \frac{x^6 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4}{3.6} A$	$\int \frac{x^7 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5}{4.7} B$	$\int \frac{x^8 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3.6}{5.8} C$
$\int \frac{x^9 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4.7}{3.6.9} A$	$\int \frac{x^{10} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5.8}{4.7.10} B$	$\int \frac{x^{11} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3.6.9}{5.8.11} C$
$\int \frac{x^{12} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4.7.10}{3.6.9.12} A$	$\int \frac{x^{13} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5.8.11}{5.7.10.13} B$	$\int \frac{x^{14} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3.6.9.12}{5.8.11.14} C$

etc.

at

at pro forma posteriori

$\int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}} = A'$	$\int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}} = B'$	$\int \frac{xx\,\partial x}{\sqrt[3]{(1-x^3)^2}} = C'$
$\int \frac{x^3\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{2} A'$	$\int \frac{x^4\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2}{4} B'$	$\int \frac{x^5\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3}{4} C'$
$\int \frac{x^6\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4}{2.5} A'$	$\int \frac{x^7\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5}{3.6} B'$	$\int \frac{x^8\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6}{4.7} C'$
$\int \frac{x^9\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4.7}{2.5.8} A'$	$\int \frac{x^{10}\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5.8}{3.6.9} B'$	$\int \frac{x^{11}\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6.9}{4.7.10} C'$
$\int \frac{x^{12}\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4.7.10}{2.5.8.11} A'$	$\int \frac{x^{13}\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5.8.11}{3.6.9.12} B'$	$\int \frac{x^{14}\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6.9.12}{4.7.10.13} C'$

vnde concludimus fore generaliter:

$$\int \frac{x^{3n}\,\partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4.\;7.\ldots(3n-2)}{3.6.\;9.\ldots\;3n} A \qquad \int \frac{x^{3n}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4.\;7.\ldots(3n-2)}{2.5.\;8.\ldots(3n-1)} A'$$

$$\int \frac{x^{3n+1}\,\partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5.\;8.\ldots(3n-1)}{4.7.10.\ldots(3n+1)} B \qquad \int \frac{x^{3n+1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5.\;8.\ldots(3n-1)}{3.6.\;9.\ldots\;3n} B'$$

$$\int \frac{x^{3n+2}\,\partial x}{\sqrt[3]{(1-x)^3}} = \frac{3.6.\;9.\ldots\;3n}{5.8.11.\ldots(3n+2)} C \qquad \int \frac{x^{3n+2}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6.\;9.\ldots\;3n}{4.7.10.\ldots(3n+1)} C'$$

notandum autem est esse $C = \frac{1}{2}$ et $C' = 1$.

Corollarium 1.

342. Hae formulae variis modis combinari possunt, vt egregia Theoremata inde oriantur, erit scilicet:

$$\int \frac{x^{3n}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n+2}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{A\,C'}{3n+1} = \frac{1}{3n+1} \int \frac{\partial x}{\sqrt[3]{(1-x)}}$$

$$\int \frac{x^{3n+1}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{A'\,B}{3n+1} = \frac{1}{3n+1} \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)}}$$

 $\int x$

$$\int\frac{x^{3n+2}\partial x}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{3n+1}\partial x}{\sqrt[3]{(1-x^3)^2}}=\frac{2B'C}{3n+2}=\frac{1}{3n+2}\int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}.$$

Corollarium 2.

343. Quia nunc ratio exponentium ad ternarium non amplius in computum ingreditur, erit generaliter:

$$\int\frac{x^{\lambda-1}\partial x}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{\lambda+1}\partial x}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{\lambda}\int\frac{\partial x}{\sqrt[3]{(1-x^3)}}$$

$$\int\frac{x^{\lambda}\partial x}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{\lambda-1}\partial x}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{\lambda}\int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$$

$$\int\frac{x^{\lambda}\partial x}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{\lambda-1}\partial x}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{\lambda}\int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}:$$

quare ex binis postremis consequimur

$$\int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{\partial x}{\sqrt[3]{(1-x^3)^2}}=\int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}$$

Corollarium 3.

344. Ponatur $x=z^n$ et $\lambda n=m$, et nostra Theoremata sequentes induent formas:

$$\int\frac{z^{m-1}\partial z}{\sqrt[3]{(1-z^{3n})}}\cdot\int\frac{z^{m+2n-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}}=\frac{1}{m}\int\frac{z^{n-1}\partial z}{\sqrt[3]{(1-z^{3n})}}$$

$$\int\frac{z^{m+n-1}\partial z}{\sqrt[3]{(1-z^{3n})}}\cdot\int\frac{z^{m-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}}=\frac{n}{m}\int\frac{z^{2n-1}\partial z}{\sqrt[3]{(1-z^{3n})}}\cdot\int\frac{z^{n-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}}$$

$$=\frac{1}{m}\int\frac{z^{2n-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}}.$$

Pro-

Problema 41.

345. Dato integrali $\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$, assignare integrale huius formulae $\int \frac{x^{m+\lambda n-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$, posito $x=1$.

Solutio.

Vt integrale sit finitum necesse est, vt m et k sint numeri positiui. Cum igitur per reductionem generalem sit

$$\int x^{m+n-1}\partial x(1-x^n)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu+n(\mu+\nu)}\int x^{m-1}\partial x(1-x^n)^{\frac{\mu}{\nu}},$$

ponatur $\nu=n$ et $\mu=k-n$, vt sit $\mu+\nu=k$, erit

$$\int \frac{x^{m+n-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \frac{m}{m+k}\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}.$$

Ponatur ergo huius formulae valor, quia datur, $=A$, haecque reductio repetita continuo dabit, posito breuitatis gratia P pro $(1-x^n)^{\frac{n-k}{n}}$,

$$\int \frac{x^{m-1}\partial x}{P} = A$$

$$\int \frac{x^{m+n-1}\partial x}{P} = \frac{m}{m+k}A$$

$$\int \frac{x^{m+2n-1}\partial x}{P} = \frac{m(m+n)}{(m+k)(m+n+k)}A$$

$$\int \frac{x^{m+3n-1}\partial x}{P} = \frac{m(m+n)(m+2n)}{(m+k)(m+n+k)(m+2n+k)}A$$

$$\int \frac{x^{m+\alpha n-1}\partial x}{P} = \frac{m(m+n)(m+2n)\ \dots\ [m+(\alpha-1)n]}{(m+k)(m+n+k)(m+2n+k)\dots[m+(\alpha-1)n+k]}A.$$

Corollarium 1.

346. Si simili modo alia formula sit $\int \frac{x^{p-1}\,\partial x}{(1-x^n)^{\frac{n-q}{n}}} = B$, posito $x=1$, at breuitatis gratia scribatur Q pro $(1-x^n)^{\frac{n-q}{n}}$, habebimus

$$\int \frac{x^{p+\alpha n-1}\,\partial x}{Q} = \frac{p\,(p+n)\,(p+2n)\;\ldots.\;[p+(\alpha-1)n]}{(p+q)(p+n+q)(p+2n+q)\ldots.[p+(\alpha-1)n+q]}B,$$

quae totidem atque illa continet factores.

Corollarium 2.

347. Statuatur nunc $p=m+k$, vt posterior numerator aequalis fiat priori denominatori, et productum harum duarum formularum est

$$\frac{m\,(m+n)\,(m+2n)\;\ldots.\;[m+(\alpha-1)\,n]}{(m+k+q)(m+n+k+q)(m+2n+k+q)\ldots.[m+(\alpha-1)n+k+q]}AB;$$

fiat porro $m+k+q=m+n$, seu $q=n-k$, erit hoc productum $=\frac{m}{m+\alpha n}AB$; ideoque

$$\int \frac{x^{m+\alpha n-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int \frac{x^{m+k+\alpha n-1}\partial x}{(1-x^n)^{\frac{k}{n}}} = \frac{m}{m+\alpha n}\int \frac{x^{m-1}\partial x}{(1+x^n)^{\frac{n-k}{n}}}\cdot\int \frac{x^{m+k-1}\partial x}{(1-x^n)^{\frac{k}{n}}}$$

quod est Theorema omni attentione dignum, cum hic non amplius opus sit, vt α sit numerus integer.

Corollarium 3.

348. Quare loco $m+\alpha n$ scribamus μ, erit:

$$\mu\int \frac{x^{\mu-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int \frac{x^{\mu+k-1}\partial x}{(1-x^n)^{\frac{k}{n}}} = m\int \frac{x^{m-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int \frac{x^{m+k-1}\partial x}{(1-x^n)^{\frac{k}{n}}}.$$

Hinc

Hinc si sumamus $m+k=n$, seu $m=n-k$, ob $\int\frac{x^{n-1}\partial x}{(1-x^n)^{\frac{k}{n}}}$

$=\frac{1-(1-x^n)^{\frac{n-k}{n}}}{n-k}=\frac{1}{n-k}$, posito $x=1$, erit

$$\int\frac{x^{\mu-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{\mu+k-1}\partial x}{(1-x^n)^{\frac{k}{n}}}=\frac{1}{\mu}\int\frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}=\frac{\pi}{\mu n\,\text{fin.}\frac{k\pi}{n}}.$$

Ac posito $x=z^\nu$, tum vero $\mu n=p$, $\nu n=q$, et $k=\lambda n$, habebitur:

$$\int\frac{z^{p-1}\partial z}{(1-z^q)^{1-\lambda}}\cdot\int\frac{z^{p+\lambda q-1}\partial z}{(1-z)^\lambda}=\frac{n}{p}\int\frac{z^{(1-\lambda)q-1}\partial z}{(1-z^q)^{1-\lambda}}.$$

Scholion 1.

349. Theoremata particuliaria, quae hinc consequuntur, ita se habebunt:

I. $n=2;\ k=1;\ \int\frac{x^{\mu-1}\partial x}{\sqrt{(1-xx)}}\cdot\int\frac{x^\mu\partial x}{\sqrt{(1-xx)}}=\frac{1}{\mu}\int\frac{\partial x}{\sqrt{(1-xx)}}=\frac{\pi}{2\mu}$

II. $n=3;\ k=1;\ \int\frac{x^{\mu-1}\partial x}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{x^\mu\partial x}{\sqrt[3]{(1-x^3)}}=\frac{1}{\mu}\int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}=\frac{2\pi}{3\mu\sqrt{3}}$

$n=3;\ k=2;\ \int\frac{x^{\mu-1}\partial x}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{\mu+1}\partial x}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{\mu}\int\frac{\partial x}{\sqrt[3]{(1-x^3)}}=\frac{2\pi}{3\mu\sqrt{3}}$

III. $n=4;\ k=1;\ \int\frac{x^{\mu-1}\partial x}{\sqrt[4]{(1-x^4)^3}}\cdot\int\frac{x^\mu\partial x}{\sqrt[4]{(1-x^4)}}=\frac{1}{\mu}\int\frac{xx\,\partial x}{\sqrt[4]{(1-x^4)^3}}=\frac{\pi}{2\mu\sqrt{2}}$

$n=4;\ k=2;\ \int\frac{x^{\mu-1}\partial x}{\sqrt{(1-x^4)}}\cdot\int\frac{x^{\mu+1}\partial x}{\sqrt{(1-x^4)}}=\frac{1}{\mu}\int\frac{x\,\partial x}{\sqrt{(1-x^4)}}=\frac{\pi}{4\mu}$

$n=4;\ k=3;\ \int\frac{x^{\mu-1}\partial x}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{x^{\mu+2}\partial x}{\sqrt[4]{(1-x^4)^3}}=\frac{1}{\mu}\int\frac{\partial x}{\sqrt[4]{(1-x^4)}}=\frac{\pi}{2\mu\sqrt{2}}$

etc. Vbi

Vbi notandum eſt, formulam $\int \frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$ ad rationalitatem reduci poſſe. Ponatur enim $\frac{x^n}{1-x^n} = z^n$, ſeu $x^n = \frac{z^n}{1+z^n}$, vnde $\frac{\partial x}{x} = \frac{\partial z}{z(1+z^n)}$. Quare cum formula noſtra ſit $= \int \left(\frac{x^n}{1-x^n}\right)^{\frac{n-k}{n}} . \frac{\partial x}{x}$, euadet ea $= \int \frac{z^{n-k-1}\partial z}{1+z^n}$, cuius integrale ita determinari debet, vt euaneſcat poſito $x = 0$ ideoque $z = 0$: tum vero poſito $x = 1$, hoc eſt $z = \infty$ dabit valorem, quo hic vtimur. Mox autem oſtendemus valorem huius integralis $\int \frac{z^{n-k-1}\partial z}{1+z^n}$, poſito $z = \infty$, ideoque et huius $\int \frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$ per angulos exprimi poſſe, quorum valores hic ſtatim appoſui. Deinde etiam notari meretur formulae $\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$ haec transformatio oriunda, poſito $1-x^n = z^n$, quae praebet $-\int \frac{z^{k-1}\partial z}{(1-z^n)^{\frac{n-m}{n}}}$ ita integranda, vt euaneſcat poſito $x = 0$ ſeu $z = 1$, tum vero ſtatui debet $x = 1$ ſeu $z = 0$. Quod eodem redit, ac ſi mutato ſigno haec formula $\int \frac{z^{k-1}\partial z}{(1-z^n)^{\frac{n-m}{n}}}$ ita integretur, vt euaneſcat, poſito $z = 0$, tum vero ponatur $z = 1$. Cum iam nihil impediat quo minus loco z ſcribamus x, habebimus hoc inſigne Theorema:

$\int x$

$$\int \frac{x^{m-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}\,\partial x}{(1-x^n)^{\frac{m-n}{n}}},$$

ita vt in huiusmodi formula exponentes m et k inter se commutare liceat, pro casu scilicet $x = 1$. Ita pro praecedente formula ad rationalitatem reducibili, vbi $m = n - k$, erit

$$\int \frac{x^{n-k-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}\,\partial x}{(1-x^n)^{\frac{k}{n}}},$$

vnde sequitur etiam fore, posito $z = \infty$,

$$\int \frac{z^{n-k-1}\,\partial z}{1+z^n} = \int \frac{z^{k-1}\,\partial z}{1+z^n}.$$

Scholion 2.

350. Hinc etiam formularum magis compositarum integralia pro casu $x = 1$, per series concinnas exprimi possunt. Cum enim in reductione superiori, posito $m + k = \mu$ seu $k = \mu - m$, fit

$$\int \frac{x^{m+n-1}\,\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}} = \frac{m}{\mu} \int \frac{x^{m-1}\,\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}},$$

si habeatur huiusmodi formula differentialis

$$\partial y = \frac{x^{m-1}\,\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}} (A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.})$$

quam ita integrari oporteat, vt y euanescat posito $x = 0$, ac requiratur valor ipsius y casu $x = 1$, erit si hoc casu fieri ponamus $\int \frac{x^{m-1}\,\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}} = O$, iste valor $=$

$$O\left(A + \frac{m}{\mu} B + \frac{m(m+n)}{\mu(\mu+n)} C + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)} D + \text{etc.}\right)$$

Viciſſim ergo propoſita hac ſerie

$$A + \frac{m}{\mu} B + \frac{m(m+n)}{\mu(\mu+n)} C + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)} D + \text{etc.}$$

eius ſumma aequabitur huic formulae integrali

$$\frac{1}{O} \int \frac{x^{m-1} \partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}} (A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.})$$

ſi poſt integrationem ponatur $x = 1$. Quod ſi ergo eueniat, vt huius ſeriei $A + Bx^n + Cx^{2n} + \text{etc.}$ ſumma aſſignari, indeque integratio abſolui queat, obtinebitur ſumma illius ſeriei.

Problema 42.

351. Integralis huius formulae $\frac{x^{m-1} \partial x}{1 + x^n}$ ita determinatum, vt poſito $x = 0$ euaneſcat, valorem caſu $x = \infty$ aſſignare.

Solutio.

Huius formulae integrale iam ſupra §. 77. exhibuimus, et quidem ita determinatum, vt poſito $x = 0$ euaneſcat, quod poſito breuitatis gratia $\frac{\pi}{n} = \omega$, ita ſe habet:

$$-\frac{2}{n} \operatorname{cof.} m\omega \, l \sqrt{(1 - 2x \operatorname{cof.} \omega + xx)} + \frac{2}{n} \operatorname{fin.} m\omega. \text{ Arc. tang. } \frac{x \operatorname{fin.} \omega}{1 - x \operatorname{cof.} \omega}$$

$$-\frac{2}{n} \operatorname{cof.} 3m\omega \, l \sqrt{(1 - 2x \operatorname{cof.} 3\omega + xx)} + \frac{2}{n} \operatorname{fin.} 3m\omega. \text{ Arc. tang. } \frac{x \operatorname{fin.} 3\omega}{1 - x \operatorname{cof.} 3\omega}$$

$$-\frac{2}{n} \operatorname{cof.} 5m\omega \, l \sqrt{(1 - 2x \operatorname{cof.} 5\omega + xx)} + \frac{2}{n} \operatorname{fin.} 5m\omega. \text{ Arc. tang. } \frac{x \operatorname{fin.} 5\omega}{1 - x \operatorname{cof.} 5\omega}$$

$$\vdots$$

$$-\frac{2}{n} \operatorname{cof.} \lambda m\omega \, l \sqrt{(1 - 2x \operatorname{cof.} \lambda\omega + xx)} + \frac{2}{n} \operatorname{fin.} \lambda m\omega. \text{ Arc. tang. } \frac{x \operatorname{fin.} \lambda\omega}{1 - x \operatorname{cof.} \lambda\omega}$$

vbi λ denotat maximum numerum imparem exponente n minorem, ac ſi n fuerit ipſe numerus impar, inſuper accedit

pars

pars $\pm\frac{1}{n}l(1+x)$, prout m fuerit vel numerus impar, vel par; illo fcilicet cafu fignum $+$, hoc vero fignum $-$ valet. Hic igitur quaeritur iftius integralis valor, qui prodit pofito $x=\infty$. Primo ergo partes logarithmos implicantes expendamus, et quia ob $x=\infty$ eft

$$l\sqrt{(1-2x\,\text{cof.}\lambda\omega+xx)}=l(x-\text{cof.}\lambda\omega)=lx+l(1-\tfrac{\text{cof.}\lambda\omega}{x})=lx,$$

ob $\frac{\text{cof.}\lambda\omega}{x}=0$; vnde partes logarithmicae praebent:

$$-\tfrac{2lx}{n}(\text{cof.}m\omega+\text{cof.}3m\omega+\text{cof.}5m\omega+\ldots+\text{cof.}\lambda m\omega)$$
$$(\pm\tfrac{lx}{n},\ \text{fi } n \text{ impar}).$$

Ponamus hanc feriem cofinuum

$$\text{cof.}m\omega+\text{cof.}3m\omega+\text{cof.}5m\omega+\ldots+\text{cof.}\lambda m\omega=s,$$

eritque per $2\,\text{fin.}m\omega$ multiplicando

$$2s\,\text{fin.}m\omega=\text{fin.}2m\omega+\text{fin.}4m\omega+\text{fin.}6m\omega\ldots+\text{fin.}(\lambda+1)m\omega$$
$$-\text{fin.}2m\omega-\text{fin.}4m\omega-\text{fin.}6m\omega,$$

vnde fit $s=\frac{\text{fin.}(\lambda+1)m\omega}{2\,\text{fin.}m\omega}$. Quare fi n fit numerus par, erit $\lambda=n-1$, ficque partes logarithmicae fiunt

$$-\tfrac{lx}{n}\cdot\tfrac{\text{fin.}mn\omega}{\text{fin.}m\omega}=-\tfrac{lx}{n}\cdot\tfrac{\text{fin.}m\pi}{\text{fin.}m\omega},\ \text{ob } n\omega=\pi.$$

At propter m numerum integrum, eft $\text{fin.}m\pi=0$, vnde hae partes euanefcunt. Sin autem fit n numerus impar, eft $\lambda=n-2$, et fumma partium logarithmicarum fit

$$-\tfrac{lx}{n}\cdot\tfrac{\text{fin.}(n-1)m\omega}{\text{fin.}m\omega}\pm\tfrac{lx}{n};$$

at $\text{fin.}(n-1)m\omega=\text{fin.}(m\pi-m\omega)=\pm\text{fin.}m\omega$, vbi fignum fuperius valet, fi m fit numerus impar, contra vero inferius, quod idem de altera ambiguitate eft tenendum, ita vt habeamus $\mp\frac{lx}{n}\cdot\frac{\text{fin.}m\omega}{\text{fin.}m\omega}\pm\frac{lx}{n}=0$. Perpetuo ergo partes logarithmicae fe mutuo tollunt; quod etiam inde eft perfpicuum, quod alioquin integrale foret infinitum, cum tamen manifefto debeat effe finitum.

Relinquuntur ergo soli anguli, quos in vnam summam colligamus; consideretur ergo Arc. tang. $\frac{x \sin. \lambda \omega}{1 - x \cos. \lambda \omega}$, qui arcus casu $x = 0$ euanescit, tum vero casu $x = \frac{1}{\cos. \lambda \omega}$ fit quadrans, vlterius ergo aucta x quadrantem superabit, donec facto $x = \infty$, eius tangens fiat $= -\frac{\sin. \lambda \omega}{\cos. \lambda \omega} = -$ tang. $\lambda \omega =$ tang. $(\pi - \lambda \omega)$, ideoque ipse arcus $= \pi - \lambda \omega$, ex quo hi arcus iunctim sumti dabunt:

$$\tfrac{2}{n}[(\pi - \omega) \sin. m\omega + (\pi - 3\omega) \sin. 3m\omega + (\pi - 5\omega) \sin. 5m\omega + \ldots \\ \ldots + (\pi - \lambda\omega) \sin. \lambda m\omega];$$

vnde duas series adipiscimur

$$\tfrac{2\pi}{n}(\sin. m\omega + \sin. 3m\omega + \sin. 5m\omega + \ldots \\ \ldots + \sin. \lambda m\omega) = \tfrac{2\pi}{n} p;$$

$$\tfrac{-2\omega}{n}(\sin. m\omega + 3 \sin. 3m\omega + 5 \sin. 5m\omega + \ldots \\ \ldots + \lambda \sin. \lambda m\omega) = \tfrac{-2\omega}{n} q;$$

quas seorsim inuestigemus, ac pro posteriori quidem cum ante habuissemus

$$\cos. m\omega + \cos. 3m\omega + \cos. 5m\omega + \ldots \\ \ldots + \cos. \lambda m\omega = s = \frac{\sin. (\lambda + 1) m\omega}{2 \sin. m\omega},$$

si angulum ω vt variabilem spectemus, differentiatio praebet

$$-m\partial\omega(\sin. m\omega + 3 \sin. 3m\omega + 5 \sin. 5m\omega + \ldots + \lambda \sin. \lambda m\omega) \\ = \frac{(\lambda+1) m\partial\omega \cos. (\lambda+1) m\omega}{2 \sin. m\omega} - \frac{m\partial\omega \sin. (\lambda+1) m\omega \cos. m\omega}{2 \sin. m\omega^2}$$

ergo

$$-q = \frac{(\lambda+1) \cos. (\lambda+1) m\omega}{2 \sin. m\omega} - \frac{\sin. (\lambda+1) m\omega \cos. m\omega}{2 \sin. m\omega^2}, \text{ seu}$$

$$-q = \frac{\lambda \cos. (\lambda+1) m\omega}{2 \sin. m\omega} - \frac{\sin. \lambda m\omega}{2 \sin. m\omega^2}.$$

Pro altera serie

$$p = \sin. m\omega + \sin. 3m\omega + \sin. 5m\omega + \ldots + \sin. \lambda m\omega,$$

multiplicemus vtrinque per $2 \sin. m\omega$, fietque

$$2p \sin. m\omega = 1 - \cos. 2m\omega - \cos. 4m\omega - \cos. 6m\omega - \ldots - \cos. (\lambda+1) m\omega$$
$$+ \cos. 2m\omega + \cos. 4m\omega + \cos. 0m\omega$$

sicque erit $p = \frac{1 - \cos. (\lambda+1) m \omega}{2 \sin. m \omega}$.

Quodsi iam fuerit n numerus par, erit $\lambda = n - 1$, indeque

$$\cos. (\lambda + 1) . m \omega = \cos. n m \omega = \cos. m \pi, \text{ et}$$
$$\sin. (\lambda + 1) m \omega = \sin. m \pi = 0, \text{ ergo}$$
$$p = \frac{1 - \cos. m \pi}{2 \sin. m \omega} \text{ et } -q = \frac{n \cos. m \pi}{2 \sin. m \omega};$$

hincque omnes arcus iunctim sumti

$$\frac{2\pi}{n} \cdot \frac{(1 - \cos. m \pi)}{2 \sin. m \omega} + \frac{2\omega}{n} \cdot \frac{n \cos. m \pi}{2 \sin. m \omega} = \frac{\pi}{n \sin. m \omega}, \text{ ob } n \omega = \pi.$$

Sit nunc n numerus impar, erit $\lambda = n - 2$, indeque

$$\cos. (\lambda + 1) m \omega = \cos. (m \pi - m \omega), \text{ et}$$
$$\sin. (\lambda + 1) m \omega = \sin. (m \pi - m \omega), \text{ seu}$$
$$\cos. (\lambda + 1) m \omega = \cos. m \pi \cos. m \omega, \text{ et}$$
$$\sin. (\lambda + 1) m \omega = - \cos. m \pi \sin. m \omega, \text{ ergo}$$
$$p = \frac{1 - \cos. m \pi \cos. m \omega}{2 \sin. m \omega} \text{ et } -q = \frac{(n-1) \cos. m \pi \cos. m \omega}{2 \sin. m \omega} + \frac{\cos. m \pi \cos. m \omega}{2 \sin. m \omega}:$$

vnde summa omnium angulorum

$$\frac{\pi (1 - \cos. m \pi \cos. m \omega)}{n \sin. m \omega} + \frac{\omega (n-1) \cos. m \pi \cos. m \omega}{n \sin. m \omega} + \frac{\omega \cos. m \pi \cos. m \omega}{n \sin. m \omega},$$

quae ob $n \omega = \pi$ reducitur ad $\frac{\pi}{n \sin. m \omega}$.

Siue ergo exponens n sit positiuus siue negatiuus, posito $x = \infty$ habemus

$$\int \frac{x^{m-1} \partial x}{1 - x^n} = \frac{\pi}{n \sin. m \omega} = \frac{\pi}{n \sin. \frac{m \pi}{n}}.$$

Corollarium 1.

352. Hinc ergo erit formula supra memorata (349)

$$\int \frac{z^{n-k-1}\partial z}{1+z^n} = \int \frac{z^{k-1}\partial z}{1+z^n} = \frac{\pi}{n\,\text{fin.}\frac{(n-k)\pi}{n}} = \frac{\pi}{n\,\text{fin.}\frac{k\pi}{n}},\ \text{posito}\ z=\infty.$$

Vnde sequitur fore etiam formulam, cui hanc aequari ostendimus:

$$\int \frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}\partial x}{(1-x^n)^{\frac{k}{n}}} = \frac{\pi}{n\,\text{fin.}\frac{k\pi}{n}},\ \text{posito}\ x=1.$$

Corollarium 2.

353. Percurramus casus simpliciores, pro vtroque formularum genere, posito $z=\infty$ et $x=1$:

$$\int \frac{\partial z}{1+zz} = \int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2\,\text{fin.}\frac{1}{2}\pi} = \frac{\pi}{2};$$

$$\int \frac{\partial z}{1+z^3} = \int \frac{z\partial z}{1+z^3} = \int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = \int \frac{x\partial x}{\sqrt[3]{(1+x^3)^2}}$$

$$= \frac{\pi}{3\,\text{fin.}\frac{1}{3}\pi} = \frac{2\pi}{3\sqrt{3}};$$

$$\int \frac{\partial z}{1+z^4} = \int \frac{zz\partial z}{1+z^4} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)}} = \int \frac{xx\partial x}{\sqrt[4]{(1-x^4)^3}}$$

$$= \frac{\pi}{4\,\text{fin.}\frac{1}{4}\pi} = \frac{\pi}{2\sqrt{2}};$$

$$\int \frac{\partial z}{1+z^6} = \int \frac{z^4\partial z}{1+z^6} = \int \frac{\partial x}{\sqrt[6]{(1-x^6)}} = \int \frac{x^4\partial x}{\sqrt[6]{(1-x^6)^5}}$$

$$= \frac{\pi}{6\,\text{fin.}\frac{1}{6}\pi} = \frac{\pi}{3}.$$

Corol-

Corollarium 3.

354. Cum ſit

$$\frac{1}{(1-x^n)^{\frac{k}{n}}} = 1 + \frac{k}{n}x^n + \frac{k(k+n)}{n.2n}x^{2n} + \frac{k(k+n)(k+2n)}{n.2n.3n}x^{3n} + \text{etc.}$$

erit per $x^{k-1}\partial x$ multiplicando, tum integrando, ac $x = 1$ ponendo

$$\frac{\pi}{n\,\text{ſin.}\frac{k\pi}{n}} = \frac{1}{k} + \frac{k}{n(k+n)} + \frac{k(k+n)}{n.2n(k+2n)} + \frac{k(k+n)(k+2n)}{n.2n.3n.(k+3n)} + \text{etc.}$$

et loco k ſcribendo $n - k$ erit quoque

$$\frac{\pi}{n\,\text{ſin.}\frac{k\pi}{n}} = \frac{1}{n-k} + \frac{n-k}{n(2n-k)} + \frac{(n-k)(2n-k)}{n.2n.(3n-k)} + \frac{(n-k)(2n-k)(3n-k)}{n.2n.\ 3n\ (4n-k)}\ \text{etc.}$$

Scholion.

355. Pro formulis quantitates tranſcendentes continentibus ſupra iam praecipuos valores, quos integralia dum variabili certus quidam valor tribuitur, recipiunt, euoluimus, ita vt non opus ſit huiusmodi formulas hic denuo examinare. Hinc autem intelligitur, eos valores integralis $\int X\,\partial x$ prae reliquis eſſe notatu dignos, ac plerumque multo ſuccinctius exprimi poſſe, qui eiusmodi valoribus variabilis x reſpondent, quibus functio X vel ſit infinita vel in nihilum abit. Ita integralia formularum $\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{\mu}{\nu}}}$ et $\int \frac{z^{m-1}\partial z}{1+z^n}$, valores prae reliquis memorabiles recipiunt, ſi fiat $x = 1$ et $z = \infty$, vbi illius denominator euaneſcit, huius vero fit infinitus. Caeterum omni attentione dignum eſt, quod hic oſtendimus, formulae integralis $\int \frac{z^{m-1}\partial z}{1+z^n}$ valorem caſu $z = \infty$ tam concinne exprimi,

vt

vt sit $\frac{\pi}{n \sin. \frac{m}{n} \pi}$, cuius demonſtratio cum per tot ambages ſit adſtructa, merito ſuſpicionem excitat, eam via multo faciliori confici poſſe, etiamſi modus nondum perſpiciatur. Id quidem manifeſtum eſt, hanc demonſtrationem ex ratione ſinuum angulorum multiplorum peti oportere; et quoniam in introductione $\sin. \frac{m}{n} \pi$ per productum infinitorum factorum expreſſi, mox videbimus, inde eandem veritatem multo facilius deduci poſſe, etiamſi ne hanc quidem viam pro maxime naturali haberi velim. Sequens autem caput huiusmodi inueſtigationi deſtinaui, quo valores integralium, quos vti in hoc capite certo quodam caſu recipiunt, per producta infinita ſeu ex innumeris factoribus conſtantia exprimere docebo; quandoquidem hinc inſignia ſubſidia in Analyſin redundant, pluraque alia incrementa inde expectari poſſunt.

CA-

CAPVT IX.

DE EVOLVTIONE INTEGRALIVM PER PRODVCTA INFINITA.

Problema 43.

356.

Valorem huius integralis $\int \frac{\partial x}{\sqrt{(1-xx)}}$, quem casu $x = 1$ recipit, in productum infinitum euoluere.

Solutio.

Quemadmodum supra formulas altiores ad simplicem reduximus, ita hic formulam $\int \frac{\partial x}{\sqrt{(1-xx)}}$ continuo ad altiores perducamus. Ita cum posito $x = 1$ sit

$$\int \frac{x^{m-1}\partial x}{\sqrt{(1-xx)}} = \frac{m+1}{m}\int \frac{x^{m+1}\partial x}{\sqrt{(1-xx)}}, \text{ erit}$$

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{2}{1}\int \frac{xx\,\partial x}{\sqrt{(1-xx)}} = \frac{2.4}{1.3}\int \frac{x^4\partial x}{\sqrt{(1-xx)}}$$

$$= \frac{2.4.6}{1.3.5}\int \frac{x^6\partial x}{\sqrt{(1-xx)}} \text{ etc.}$$

vnde concludimus fore indefinite:

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{2.4.6.8 \ldots\ldots 2i}{1.3.5.7 \ldots\ldots (2i-1)}\int \frac{x^{2i}\partial x}{\sqrt{(1-xx)}}$$

atque adeo etiam si pro i sumatur numerus infinitus. Nunc simili modo a formula $\int \frac{x\,\partial x}{\sqrt{(1-xx)}}$ ascendamus, reperiemusque

$$\int \frac{x\,\partial x}{\sqrt{(1-xx)}} = \frac{3.5.7.9\ldots(2i+1)}{2.4.6.8\ldots\ldots 2i} \int \frac{x^{2i+1}\,\partial x}{\sqrt{(1-xx)}},$$

atque obferuo, fi i fit numerus infinitus, formulas iftas

$$\int \frac{x^{2i}\,\partial x}{\sqrt{(1-xx)}} \text{ et } \int \frac{x^{2i+1}\,\partial x}{\sqrt{(1-xx)}}$$

rationem aequalitatis eſſe habituras. Ex reductione enim principali perfpicuum eft, fi m fit numerus infinitus, fore

$$\int \frac{x^{m-1}\,\partial x}{\sqrt{(1-xx)}} = \int \frac{x^{m+1}\,\partial x}{\sqrt{(1-xx)}} = \int \frac{x^{m+3}\,\partial x}{\sqrt{(1-xx)}}$$

atque adeo in genere $\int \frac{x^{m+\mu}\,\partial x}{\sqrt{(1-xx)}} = \int \frac{x^{m+\nu}\,\partial x}{\sqrt{(1-xx)}}$ quantumuis magna fuerit differentia inter μ et ν, modo finita. Cum igitur fit $\int \frac{x^{2i}\,\partial x}{\sqrt{(1-xx)}} = \int \frac{x^{2i+1}\,\partial x}{\sqrt{(1-xx)}}$, fi ponamus:

$$\frac{2.4.6.\ldots\ldots 2i}{1.3.5.\ldots\ldots(2i-1)} = \mathrm{M} \text{ et } \frac{3.5.7.9.\ldots\ldots(2i+1)}{2.4.6.8.\ldots\ldots 2i} = \mathrm{N}, \text{ erit}$$

$$\int \frac{\partial x}{\sqrt{(1-xx)}} : \int \frac{x\,\partial x}{\sqrt{(1-xx)}} = \mathrm{M} : \mathrm{N} = \frac{\mathrm{M}}{\mathrm{N}} : 1, \text{ pofito } x = 1.$$

At eft $\int \frac{x\,\partial x}{\sqrt{(1-xx)}} = 1$ et $\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2}$,

vnde colligitur $\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\mathrm{M}}{\mathrm{N}}$, quia producta M et N ex aequali factorum numero conftant, fi primum factorem $\frac{2}{1}$ producti M per primum factorem $\frac{3}{2}$ producti N, fecundum $\frac{4}{3}$ illius, per fecundum $\frac{5}{4}$ huius et ita porro diuidamus, fiet

$$\frac{\mathrm{M}}{\mathrm{N}} = \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdot \frac{8.8}{7.9} \cdot \text{etc.}$$

vnde obtinemus pro cafu $x = 1$, per productum infinitum,

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdot \frac{8.8}{7.9} \cdot \text{etc.} = \frac{\pi}{2}.$$

Corollarium 1.

357. Pro valore ergo ipfius π idem productum infinitum elicuimus, quod olim iam Wallifius inuenerat, et cuius

ius veritatem in Introductione confirmauimus, diuersissimis viis incedentes, erit itaque

$$\pi = 2 \cdot \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdot \frac{8 \cdot 8}{7 \cdot 9} \cdot \text{etc.}$$

Corollarium 2.

358. Nihil interest, quonam ordine singuli factores in hoc producto disponantur, dummodo nulli relinquantur. Ita aliquot ab initio seorsim sumendo, reliqui ordine debito disponi possunt; veluti

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{8 \cdot 10}{9 \cdot 9} \cdot \text{etc. vel}$$

$$\frac{\pi}{2} = \frac{2 \cdot 4}{1 \cdot 3} \times \frac{2 \cdot 6}{3 \cdot 5} \cdot \frac{4 \cdot 8}{5 \cdot 7} \cdot \frac{6 \cdot 10}{7 \cdot 9} \cdot \frac{8 \cdot 12}{9 \cdot 11} \cdot \text{etc. vel}$$

$$\frac{\pi}{2} = \frac{2}{3} \times \frac{2 \cdot 4}{1 \cdot 5} \cdot \frac{4 \cdot 6}{3 \cdot 7} \cdot \frac{6 \cdot 8}{5 \cdot 9} \cdot \frac{8 \cdot 10}{7 \cdot 11} \cdot \text{etc. vel}$$

$$\frac{\pi}{2} = \frac{2 \cdot 4}{3 \cdot 5} \times \frac{2 \cdot 6}{1 \cdot 7} \cdot \frac{4 \cdot 8}{3 \cdot 9} \cdot \frac{6 \cdot 10}{5 \cdot 11} \cdot \frac{8 \cdot 12}{7 \cdot 13} \cdot \text{etc.}$$

Scholion.

359. Fundamentum ergo huius euolutionis in hoc consistit, quod valor integralis $\int \frac{x^{i+\alpha} \partial x}{\sqrt{(1 - xx)}}$, denotante i numerum infinitum, idem sit, vtcunque numerus finitus α varietur. Atque hoc quidem ex reductione

$$\int \frac{x^{i-1} \partial x}{\sqrt{(1 - xx)}} = \frac{i+1}{i} \int \frac{x^{i+1} \partial x}{\sqrt{(1 - xx)}}$$

manifestum est, si pro α valores binario differentes assumantur. Deinde autem nullum est dubium, quin hoc integrale $\int \frac{x^{i+1} \partial x}{\sqrt{(1 - xx)}}$ inter haec $\int \frac{x^{i} \partial x}{\sqrt{(1 - xx)}}$ et $\int \frac{x^{i+2} \partial x}{\sqrt{(1 - xx)}}$, quasi limites contineatur, qui cum sint inter se aequales, necesse est omnes formulas intermedias iisdem quoque esse aequales

quales. Atque hoc latius patet ad formulas magis complicatas, ita vt denotante i numerum infinitum fit

$$\int \frac{x^{i+\alpha}\,\partial x}{(1-x^n)^k} = \int \frac{x^i\,\partial x}{(1-x^n)^k}.$$

Cum enim fit

$$\int \frac{x^{m+n-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \frac{m}{m+k} \int \frac{x^{m-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}}$$

hae formulae pofito $m = \infty$ funt aequales; vnde illarum quoque aequalitas cafibus, quibus $\alpha = n$, vel $\alpha = 2n$, vel $\alpha = 3n$ etc. perfpicitur; fin autem α medium quempiam valorem teneat formulae, ipfius quoque valor medium quoddam tenere debet inter valores aequales, ideoque ipfis erit aequalis. Hoc igitur principio ftabilito fequens problema refoluere poterimus.

Problema 44.

360. Rationem horum duorum integralium

$$\int x^{m-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}} \text{ et } \int x^{\mu-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}},$$

cafu $x = 1$, per productum infinitorum factorum exprimere.

Solutio.

Cum fit

$$\int x^{m-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}} = \frac{m+k}{m} \int x^{m+n-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}},$$

cafu $x = 1$, valor iftius integralis ad integrale infinite remotum reducetur hoc modo:

$$\int x^{m-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}}$$

$$= \frac{(m+k)(m+k+n)(m+k+2n)\ldots\ldots(m+k+in)}{m\ \ (m+n)\ (m+2n)\ldots\ldots(m+in)} \int x^{m+in+n-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}},$$

vbi i numerum infinitum denotare affumimus. Simili autem modo

modo pro altera formula propofita erit

$$\int x^{\mu-1}\partial x\,(1-x^n)^{\frac{k-n}{n}}$$
$$=\frac{(\mu+k)(\mu+k+n)(\mu+k+2n)\ldots\ldots(\mu+k+in)}{\mu\quad(\mu+n)\quad(\mu+2n)\ldots\ldots(\mu+in)}\int x^{\mu+in+n-1}\partial x\,(1-x^n)^{\frac{k-n}{n}},$$

atque hae poftremae formulae integrales ob exponentes infinitos, aequales erunt, non obftante inaequalitate numerorum m et μ: tum vero bina haec producta infinita pari factorum numero conftant. Quare fi finguli per fingulos, hoc eft primus per primum, fecundus per fecundum diuidantur, ratio binorum integralium propofitorum ita exprimetur:

$$\frac{\int x^{m-1}\partial x\,(1-x^n)^{\frac{k-n}{n}}}{\int x^{\mu-1}\partial x\,(1-x^n)^{\frac{k-n}{n}}}=\frac{\mu(m+k)}{m(\mu+k)}\cdot\frac{(\mu+n)(m+k+n)}{(m+n)(\mu+k+n)}\cdot\frac{(\mu+2n)(m+k+2n)}{(m+2n)(\mu+k+2n)}\text{ etc.}$$

fi quidem ambo integralia ita determinentur, vt pofito $x=0$ euanefcant, tum vero ftatuatur $x=1$; litteris autem m, μ, n, k numeros pofitiuos denotari neceffe eft.

Corollarium 1.

361. Si differentia numerorum m et μ aequetur multiplo ipfius n, in producto inuento infiniti factores fe deftruunt, relinqueturque factorum numerus finitus, vti fi $\mu=m+n$ habebitur:

$$\frac{(m+n)(m+k)}{m(m+k+n)}\cdot\frac{(m+2n)(m+k+n)}{(m+n)(m+k+2n)}\cdot\frac{(m+3n)(m+k+2n)}{(m+2n)(m+k+3n)}\text{ etc.}$$

quod reducitur ad $\frac{m+k}{m}$.

Corollarium 2.

362. Valor autem illius producti neceffario eft finitus, id quod tam ex formulis integralibus, quarum rationem exprimit, patet, quam inde, quod in fingulis factoribus nu-

meratores et denominatores sunt alternatim maiores et minores.

Corollarium 3.

363. Si ponamus $m = 1$, $\mu = 3$, $n = 4$ et $k = 2$, erit

$$\frac{\int \frac{\partial x}{\sqrt{(1-x^4)}}}{\int \frac{x x \partial x}{\sqrt{(1-x^4)}}} = \frac{3 \cdot 3}{1 \cdot 5} \cdot \frac{7 \cdot 7}{5 \cdot 9} \cdot \frac{11 \cdot 11}{9 \cdot 13} \cdot \frac{15 \cdot 15}{13 \cdot 17} \text{ etc.}$$

supra autem inuenimus productum harum binarum formularum esse $= \frac{\pi}{4}$.

Problema 45.

364. Valorem huius integralis $\int x^{m-1} \partial x (1-x^n)^{\frac{k-n}{n}}$, quem posito $x = 1$ recipit, per productum infinitum exprimere.

Solutio.

Cum in problemate praecedente ratio huius integralis ad hoc alterum $\int x^{\mu-1} \partial x (1-x^n)^{\frac{k-n}{n}}$ per productum infinitum sit assignata, in hoc exponens μ ita accipiatur, vt integrale exhiberi possit. Capiatur ergo $\mu = n$, et integrale fit $=$

$$C - \frac{1}{k}(1-x^n)^{\frac{k}{n}} = \frac{1-(1-x^n)^{\frac{k}{n}}}{k}$$

ita determinatum, vt posito $x = 0$ euanescat: ponatur nunc, vt conditio postulat, $x = 1$, et quia hoc integrale erit $= \frac{1}{k}$, habebimus formulae propositae integrale casu $x = 1$, ita expressum

$$\int x^{m-1} \partial x (1-x^n)^{\frac{k-n}{n}} = \frac{1}{k} \cdot \frac{n(m+k)}{m(k+n)} \cdot \frac{2n(m+k+n)}{(m+n)(k+2n)} \cdot \frac{3n(m+k+2n)}{(m+2n)(k+3n)} \text{ etc.}$$

quod singulos factores partiendo ita repraesentari potest

$\int x$

$$\int x^{m-1}\partial x(1-x^n)^{\frac{k-n}{n}} = \frac{n}{mk}\cdot\frac{2n(m+k)}{(m+n)(k+n)}\cdot\frac{3n(m+k+n)}{(m+2n)(k+2n)}\cdot\frac{4n(m+k+2n)}{(m+3n)(k+3n)}\text{ etc.}$$

Corollarium 1.

365. Cum in hac expressione litterae m et k sint permutabiles, sequitur etiam, haec integralia posito $x=1$, inter se esse aequalia:

$$\int x^{m-1}\partial x(1-x^n)^{\frac{k-n}{n}} = \int^{k-1}\partial x(1-x^n)^{\frac{m-n}{n}}$$

quam aequalitatem iam supra §. 349. elicuimus.

Corollarium 2.

366. Cum formulae nostrae valor, si $m=n-k$, aequalis sit valori huius $\int\frac{z^{k-1}\partial z}{1+z^n}$ posito $z=\infty$, si ob $m+k=n$ statuamus $m=\frac{n+\alpha}{2}$ et $k=\frac{n-\alpha}{2}$, habebimus:

$$\int\frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n+\alpha}{2n}}} = \frac{x^{k-1}\partial x}{(1-x^n)^{\frac{n-\alpha}{2n}}} = \int\frac{z^{k-1}\partial z}{1+z^n} = \int\frac{z^{m-1}\partial z}{1+z^n}$$

$$= \frac{4n}{nn-\alpha\alpha}\cdot\frac{2.4nn}{9nn-\alpha\alpha}\cdot\frac{4.6nn}{25nn-\alpha\alpha}\cdot\frac{6.8nn}{49nn-\alpha\alpha}\text{ etc.}$$

Quod productum etiam hoc modo exponi potest

$$\frac{2}{n-\alpha}\cdot\frac{2n.2n}{(n+\alpha)(3n-\alpha)}\cdot\frac{4n.4n}{(3n+\alpha)(5n-\alpha)}\cdot\frac{6n.6n}{(5n+\alpha)(7n-\alpha)}\text{ etc.}$$

quod ergo etiam exprimit valorem ipsius $\frac{\pi}{n\sin.\frac{m\pi}{n}} = \frac{\pi}{n\cos.\frac{\alpha\pi}{2n}}$ per §. 352.

Corollarium 3.

367. Vel si simpliciter ponamus $k=n-m$, fiet

$\int x'$

$$\int\frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m}{n}}}=\int\frac{x^{n-m-1}\partial x}{(1-x^n)^{\frac{n-m}{n}}}=\int\frac{z^{m-1}\partial z}{1+z^n}=\int\frac{z^{n-m-1}\partial z}{1+z^n}$$

$$=\frac{1}{n-m}\cdot\frac{nn}{m(2n-m)}\cdot\frac{4nn}{(n+m)(3n-m)}\cdot\frac{9nn}{(2n+m)(4n-m)}\text{ etc.}$$

quae ex forma primum inuenta oritur. Haec ergo aequalitas subsistit, si ponatur $x=1$ et $z=\infty$.

Scholion 1.

368. In Introductione autem pro multiplicatione angulorum inueneram

$$\text{fin.}\frac{m\pi}{n}=\frac{m\pi}{n}\left(1-\frac{mm}{nn}\right)\left(1-\frac{mm}{4nn}\right)\left(1-\frac{mm}{9nn}\right)\left(1-\frac{mm}{16nn}\right)\text{ etc.}$$

et cum fin. $\frac{(n-m)\pi}{n}=$ fin. $\frac{m\pi}{n}$, ob $n-m=k$, erit etiam

$$\text{fin.}\frac{m\pi}{n}=\frac{k\pi}{n}\left(1-\frac{kk}{nn}\right)\left(1-\frac{kk}{4nn}\right)\left(1-\frac{kk}{9nn}\right)\left(1-\frac{kk}{16nn}\right)\text{ etc.}$$

quae reducitur ad hanc formam

$$\text{fin.}\frac{m\pi}{n}=\frac{k\pi}{n}\cdot\frac{(n-k)(n+k)}{nn}\cdot\frac{(2n-k)(2n+k)}{4nn}\cdot\frac{(3n-k)(3n+k)}{9nn}\text{ etc.}$$

et pro k suo valore restituto

$$\text{fin.}\frac{m\pi}{n}=\frac{\pi}{n}(n-m)\cdot\frac{m(2n-m)}{nn}\cdot\frac{(n+m)(3n-m)}{4nn}\cdot\frac{(2n+m)(4n-m)}{9nn}\text{ etc.}$$

vnde manifesto pro $\frac{\pi}{n\,\text{fin.}\frac{m\pi}{n}}$ idem reperitur productum, quod valorem nostrorum integralium exprimit, sicque nouam habemus demonstrationem pro Theoremate illo eximio supra per multas ambages euicto, esse

$$\int\frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m}{n}}}=\int\frac{x^{n-m-1}\partial x}{(1-x^n)^{\frac{n-m}{n}}}=\int\frac{z^{m-1}\partial z}{1+z^n}=\int\frac{z^{n-m-1}\partial z}{1+z^n}$$

$$=\frac{\pi}{n\,\text{fin.}\frac{m\pi}{n}}.$$

Scho-

Scholion 2.

369. Quo noſtra formula latius pateat, ponamus $\frac{k}{n} = \frac{\mu}{\nu}$ ſeu $k = \frac{\mu n}{\nu}$, et nanciſcemur $\int x^{m-1}\partial x\,(1-x^n)^{\frac{\mu}{\nu}-1}$

$$= \frac{\nu}{m\mu}\cdot\frac{2(m\nu+n\mu)}{(m+n)(\mu+\nu)}\cdot\frac{3[m\nu+n(\mu+\nu)]}{(m+2n)(\mu+2\nu)}\cdot\frac{4[m\nu+n(\mu+2\nu)]}{(m+3n)(\mu+3\nu)}\cdot\text{ etc.}$$

$$= \frac{\nu}{m\mu}\cdot\frac{2(m\nu+n\mu)}{(m+n)(\mu+\nu)}\cdot\frac{3(m\nu+n\mu+n\nu)}{(m+2n)(\mu+2\nu)}\cdot\frac{4(m\nu+n\mu+2n\nu)}{(m+3n)(\mu+3\nu)}\cdot\frac{5(m\nu+n\mu+3n\nu)}{(m+4n)(\mu+4\nu)}\text{ etc.}$$

in qua expreſſione litterae m, n et μ, ν ſunt permutabiles, praeterquam in primo factore, qui cum reliquis lege continuitatis non connectitur; ac ſi per n multiplicemus, permutabilitas erit perfecta, vnde concludimus fore

$$n\int x^{m-1}\partial x\,(1-x^n)^{\frac{\mu}{\nu}-1} = \nu\int x^{\mu-1}\partial x\,(1-x^\nu)^{\frac{m}{n}-1}$$

quae aequalitas caſu $\nu = n$ ad ſupra obſeruatam reducitur. Caeterum iuuabit caſus praecipuos perpendiſſe, quos ex valoribus μ et ν deſumamus.

Exemplum 1.

370. *Sit* $\mu = 1$ et $\nu = 2$, *fietque*

$$\int\frac{x^{m-1}\partial x}{\sqrt{(1-x^n)}} = \frac{2}{m}\cdot\frac{2(2m+n)}{3(m+n)}\cdot\frac{3(2m+3n)}{5(m+2n)}\cdot\frac{4(2m+5n)}{7(m+3n)}\text{ etc.}$$

$$= \frac{2}{n}\int\frac{\partial x}{\sqrt[n]{(1-x^2)^{n-m}}}$$

quae expreſſio ita commodius repraeſentatur:

$$\int\frac{x^{m-1}\partial x}{\sqrt{(1-x^n)}} = \frac{2}{m}\cdot\frac{4(2m+n)}{3(2m+2n)}\cdot\frac{6(2m+3n)}{5(2m+4n)}\cdot\frac{8(2m+5n)}{7(2m+6n)}\text{ etc.}$$

vnde ſequentes caſus ſpecialiſſimi deducuntur:

$$\int\frac{\partial x}{\sqrt{(1-xx)}} = 2\cdot\frac{2.4}{3.3}\cdot\frac{4.6}{5.5}\cdot\frac{6.8}{7.7}\text{ etc.} \qquad = \int\frac{\partial x}{\sqrt{(1-xx)}}$$

$$\int \frac{\partial x}{\sqrt{(1-x^3)}} = 2 \cdot \frac{4.5}{3.4} \cdot \frac{6.11}{5.14} \cdot \frac{8.17}{7.20} \cdot \frac{10.23}{9.26} \text{ etc.} = \frac{2}{3} \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$$

$$\int \frac{x \partial x}{\sqrt{(1-x^3)}} = 1 \cdot \frac{4.7}{3.10} \cdot \frac{6.13}{5.16} \cdot \frac{8.19}{7.22} \cdot \frac{10.25}{9.28} \text{ etc.} = \frac{2}{3} \int \frac{\partial x}{\sqrt[3]{(1-x^3)}}$$

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} = 2 \cdot \frac{4.3}{3.5} \cdot \frac{6.7}{5.9} \cdot \frac{8.11}{7.13} \cdot \frac{10.15}{9.17} \text{ etc.} = \frac{1}{2} \int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}}$$

$$\int \frac{x \partial x}{\sqrt{(1-x^4)}} = 1 \cdot \frac{4.4}{3.6} \cdot \frac{6.8}{5.10} \cdot \frac{8.12}{7.14} \cdot \frac{10.16}{9.18} \text{ etc.} = \frac{1}{2} \int \frac{\partial x}{\sqrt[4]{(1-x^4)^2}}$$

$$\text{fiue} = 1 \cdot \frac{2.4}{3.3} \cdot \frac{4.6}{5.5} \cdot \frac{6.8}{7.7} \cdot \frac{8.10}{9.9} \text{ etc.}$$

$$\int \frac{x x \partial x}{\sqrt{(1-x^4)}} = \frac{2}{3} \cdot \frac{4.5}{3.7} \cdot \frac{6.9}{5.11} \cdot \frac{8.13}{7.15} \cdot \frac{10.17}{9.19} \text{ etc.} = \frac{1}{2} \int \frac{\partial x}{\sqrt[4]{(1-x^4)}}$$

$$\int \frac{x^3 \partial x}{\sqrt{(1-x^4)}} = \frac{2}{4} \cdot \frac{4.6}{3.8} \cdot \frac{6.10}{5.12} \cdot \frac{8.14}{7.16} \cdot \frac{10.18}{9.20} \text{ etc.} = \frac{1}{2}.$$

Exemplum 2.

271. *Sit* $\mu = 1$ *et* $\nu = 3$, *fietque*

$$\int \frac{x^{m-1} \partial x}{\sqrt[3]{(1-x^n)^2}} = \frac{3}{m} \cdot \frac{2(3m+n)}{4(m+n)} \cdot \frac{3(3m+4n)}{7(m+2n)} \cdot \frac{4(3m+7n)}{10(m+3n)} \text{ etc.}$$

$$= \frac{3}{n} \int \frac{\partial x}{\sqrt[n]{(1-x^3)^{n-m}}} :$$

vnde ſequentes caſus ſpecialiſſimi deducuntur:

$$\int \frac{\partial x}{\sqrt[3]{(1-x^2)^2}} = \frac{3}{1} \cdot \frac{2.5}{4.3} \cdot \frac{3.11}{7.5} \cdot \frac{4.17}{10.7} \cdot \frac{5.23}{13.9} \text{ etc.} = \frac{3}{2} \int \frac{\partial x}{\sqrt{(1-x^3)}}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3}{1} \cdot \frac{2.6}{4.4} \cdot \frac{3.15}{7.7} \cdot \frac{4.24}{10.10} \cdot \frac{5.33}{13.13} \text{ etc.} = \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$$

$$\text{fiue} = \frac{3}{1} \cdot \frac{2.6}{4.4} \cdot \frac{5.9}{7.7} \cdot \frac{8.12}{10.10} \cdot \frac{11.15}{13.13} \text{ etc.}$$

$$\int \frac{x \partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3}{2} \cdot \frac{2.9}{4.5} \cdot \frac{3.18}{7.8} \cdot \frac{4.27}{10.11} \cdot \frac{5.36}{13.14} \text{ etc.} = \int \frac{\partial x}{\sqrt[3]{(1-x^3)}}$$

$$\text{fiue} = \frac{3}{2} \cdot \frac{3.6}{4.5} \cdot \frac{6.9}{7.8} \cdot \frac{9.12}{10.11} \cdot \frac{12.15}{13.14} \text{ etc.}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^4)^2}} = \frac{3}{1} \cdot \frac{2.7}{4.5} \cdot \frac{3.19}{7.9} \cdot \frac{4.31}{10.13} \cdot \frac{5.43}{13.17} \text{ etc.} = \frac{3}{4} \int \frac{\partial x}{\sqrt[4]{(1-x^3)^3}}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^4)^2}} = 1 \cdot \frac{2.13}{4.7} \cdot \frac{3.25}{7.11} \cdot \frac{4.37}{10.15} \cdot \frac{5.49}{13.19} \text{ etc.} = \frac{3}{4} \int \frac{\partial x}{\sqrt[4]{(1-x^3)}}$$

Ex-

Exemplum 3.

372. *Sit* $\mu = 2$ *et* $\nu = 3$, *fietque*

$$\int \frac{x^{m-1}\partial x}{\sqrt[3]{(1-x^n)}} = \frac{3}{2m} \cdot \frac{2(3m+2n)}{5(m+n)} \cdot \frac{3(3m+5n)}{8(m+2n)} \cdot \frac{4(3m+8n)}{11(m+3n)} \text{ etc.}$$

$$= \frac{3}{n} \int \frac{x\,\partial x}{\sqrt[n]{(1-x^3)^{n-m}}} :$$

vnde sequentes casus speciales deducuntur:

$$\int \frac{\partial x}{\sqrt[3]{(1-x^2)}} = \frac{3}{2} \cdot \frac{2.\ 7}{5.\ 3} \cdot \frac{3.\ 13}{8.\ 5} \cdot \frac{4.\ 19}{11.\ 7} \cdot \frac{5.\ 25}{14.\ 9} \cdot \text{etc.} = \frac{3}{2} \int \frac{x\,\partial x}{\sqrt{(1-x^3)}}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = \frac{3}{3} \cdot \frac{2.\ 9}{5.\ 4} \cdot \frac{3.\ 18}{8.\ 7} \cdot \frac{4.\ 27}{11.\ 10} \cdot \frac{5.\ 36}{14.\ 13} \cdot \text{etc.} = \int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}$$

siue

$$= \frac{3.\ 3}{2.\ 4} \cdot \frac{6.\ 6}{5.\ 7} \cdot \frac{9.\ 9}{8.\ 10} \cdot \frac{12.\ 12}{11.\ 13} \text{ etc.}$$

$$\int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)}} = \frac{3}{4} \cdot \frac{2.\ 12}{5.\ 5} \cdot \frac{3.\ 21}{8.\ 8} \cdot \frac{4.\ 30}{11.\ 11} \cdot \frac{5.\ 39}{14.\ 14} \cdot \text{etc.} = \int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)}}$$

siue

$$= \frac{3}{4} \cdot \frac{4.\ 6}{5.\ 5} \cdot \frac{7.\ 9}{8.\ 8} \cdot \frac{10.\ 12}{11.\ 11} \cdot \frac{13.\ 15}{14.\ 14} \text{ etc.}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^4)}} = \frac{3}{2} \cdot \frac{2.\ 11}{5.\ 5} \cdot \frac{3.\ 23}{8.\ 9} \cdot \frac{4.\ 35}{11.\ 13} \cdot \frac{5.\ 37}{14.\ 17} \cdot \text{etc.} = \frac{3}{4} \int \frac{x\,\partial x}{\sqrt[4]{(1-x^3)^3}}$$

$$\int \frac{x^2\,\partial x}{\sqrt[3]{(1-x^4)}} = \frac{1}{2} \cdot \frac{2.\ 17}{5.\ 7} \cdot \frac{3.\ 29}{8.\ 11} \cdot \frac{4.\ 41}{11.\ 15} \cdot \frac{5.\ 53}{14.\ 19} \cdot \text{etc.} = \frac{3}{4} \int \frac{x\,\partial x}{\sqrt[4]{(1-x^3)}}.$$

Exemplum 4.

373. *Sit* $\mu = 1$ *et* $\nu = 4$, *fietque*

$$\int \frac{x^{m-1}\partial x}{\sqrt[4]{(1-x^n)^3}} = \frac{4}{m} \cdot \frac{2(4m+n)}{5(m+n)} \cdot \frac{3(4m+5n)}{9(m+2n)} \cdot \frac{4(4m+9n)}{13(m+3n)} \text{ etc.}$$

$$= \frac{4}{n} \int \frac{\partial x}{\sqrt[n]{(1-x^4)^{n-m}}} :$$

vnde sequentes casus speciales prodeunt:

 $\int \partial x$

$$\int \frac{\partial x}{\sqrt[4]{(1-x^3)^3}} = \frac{4}{1} \cdot \frac{2 \cdot 6}{5 \cdot 3} \cdot \frac{3 \cdot 14}{9 \cdot 5} \cdot \frac{4 \cdot 22}{13 \cdot 7} \cdot \frac{5 \cdot 30}{17 \cdot 9} \cdot \text{etc.} = 2 \int \frac{\partial x}{\sqrt{(1-x^4)}}$$

seu $$= \frac{4}{1} \cdot \frac{4 \cdot 3}{3 \cdot 5} \cdot \frac{6 \cdot 7}{5 \cdot 9} \cdot \frac{8 \cdot 11}{7 \cdot 13} \cdot \frac{10 \cdot 15}{9 \cdot 17} \text{ etc.}$$

$$\int \frac{\partial x}{\sqrt[4]{(1-x^3)^3}} = \frac{4}{1} \cdot \frac{2 \cdot 7}{5 \cdot 4} \cdot \frac{3 \cdot 19}{9 \cdot 7} \cdot \frac{4 \cdot 31}{13 \cdot 10} \cdot \frac{5 \cdot 43}{17 \cdot 13} \cdot \text{etc.} = \frac{4}{3} \int \frac{\partial x}{\sqrt[3]{(1-x^4)^2}}$$

$$\int \frac{x \partial x}{\sqrt[4]{(1-x^3)^3}} = \frac{2}{1} \cdot \frac{2 \cdot 11}{5 \cdot 5} \cdot \frac{3 \cdot 23}{9 \cdot 8} \cdot \frac{4 \cdot 35}{13 \cdot 11} \cdot \frac{5 \cdot 47}{17 \cdot 14} \cdot \text{etc.} = \frac{4}{3} \int \frac{\partial x}{\sqrt[3]{(1-x^4)}}$$

$$\int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} = \frac{4}{1} \cdot \frac{2 \cdot 8}{5 \cdot 5} \cdot \frac{3 \cdot 24}{9 \cdot 9} \cdot \frac{4 \cdot 40}{13 \cdot 13} \cdot \frac{5 \cdot 56}{17 \cdot 17} \cdot \text{etc.} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}}$$

seu $$= \frac{4}{1} \cdot \frac{4 \cdot 4}{5 \cdot 5} \cdot \frac{6 \cdot 12}{9 \cdot 9} \cdot \frac{8 \cdot 20}{13 \cdot 13} \cdot \frac{10 \cdot 28}{17 \cdot 17} \text{ etc.}$$

seu $$= \frac{4}{1} \cdot \frac{2 \cdot 8}{5 \cdot 5} \cdot \frac{6 \cdot 12}{9 \cdot 9} \cdot \frac{10 \cdot 16}{13 \cdot 13} \cdot \frac{14 \cdot 20}{17 \cdot 17} \text{ etc.}$$

$$\int \frac{x x \partial x}{\sqrt[4]{(1-x^4)^3}} = \frac{4}{3} \cdot \frac{2 \cdot 16}{5 \cdot 7} \cdot \frac{3 \cdot 32}{9 \cdot 11} \cdot \frac{4 \cdot 48}{13 \cdot 15} \cdot \frac{5 \cdot 64}{17 \cdot 19} \cdot \text{etc.} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)}}$$

seu $$= \frac{4}{3} \cdot \frac{4 \cdot 8}{5 \cdot 7} \cdot \frac{6 \cdot 16}{9 \cdot 11} \cdot \frac{8 \cdot 24}{13 \cdot 15} \cdot \frac{10 \cdot 32}{17 \cdot 19} \text{ etc.}$$

seu $$= \frac{4}{3} \cdot \frac{4 \cdot 8}{5 \cdot 7} \cdot \frac{8 \cdot 12}{9 \cdot 11} \cdot \frac{12 \cdot 16}{13 \cdot 15} \cdot \frac{16 \cdot 20}{17 \cdot 19} \text{ etc.}$$

Atque in his et praecedentibus iam casus $\mu = 3$ et $\nu = 4$ est contentus.

Scholion.

374. Caeterum hae formulae, in quas litteras μ et ν introduxi, latius non patent quam primum consideratae, series enim pendent a binis fractionibus $\frac{m}{n}$ et $\frac{\mu}{\nu}$, quae cum semper ad communem denominatorem reuocari queant, formulas

$$\int \frac{x^{m-1} \partial x}{\sqrt[n]{(1-x^n)^{n-k}}} = \int \frac{x^{k-1} \partial x}{\sqrt[n]{(1-x^n)^{n-m}}}$$

perpendisse sufficiet. Cum igitur earum valor casu $x = 1$ aequetur huic producto

$$\frac{1}{k} \cdot \frac{n(m+k)}{m(k+n)} \cdot \frac{2n(m+k+n)}{(m+n)(k+2n)} \cdot \frac{3n(m+k+2n)}{(m+2n)(k+3n)} \cdot \text{etc.}$$

si

ſi in ſingulis membris factores numeratorum permutemus, et membra aliter partiamur, idem productum hanc induet formam

$$\frac{m+k}{mk}\cdot\frac{n(m+k+n)}{(m+n)(k+n)}\cdot\frac{2n(m+k+3n)}{(m+2n)(k+2n)}\cdot\frac{3n(m+k+3n)}{(m+3n)(k+3n)}\text{ etc.}$$

quae ad memoriam magis accommodata videtur. Simili modo cum ſit:

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}=\int\frac{x^{q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

$$=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\frac{2n(p+q+3n)}{(p+3n)(q+3n)}\text{ etc.}$$

illam formam per hanc diuidendo, erit

$$\frac{\int x^{m-1}\partial x(1-x^n)^{\frac{k-n}{n}}}{\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}}}$$

$$=\frac{pq(m+k)}{mk(p+q)}\cdot\frac{(p+n)(q+n)(m+k+n)}{(m+n)(k+n)(p+q+n)}\cdot\frac{(p+2n)(q+2n)(m+k+2n)}{(m+2n)(k+2n)(p+q+2n)}\text{ etc.}$$

cuius omnia membra eadem lege continentur. Hinc autem eximiae comparationes huiusmodi formularum deduci poſſunt, quae quo facilius commemorari queant, breuitatis cauſa ſequenti ſcriptionis compendio vtar.

Definitio.

375. Formulae integralis $\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}}$ valorem, quem poſito $x=1$ recipit, breuitatis gratia hoc ſigno $\left(\frac{p}{q}\right)$ indicemus, vbi quidem exponentem n, quem in comparatione plurium huiusmodi formularum eundem eſſe aſſumo, ſubintelligi oportet.

Corollarium 1.

376. Primum igitur patet eſſe $\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right)$, et vtramque formulam eſſe

$$=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\text{ etc.}$$

 quo-

quorum membrorum progreſſio eſt manifeſta, dum ſinguli factores tam numeratoris quam denominatoris continuo eodem numero n augentur, ita vt ex cognito primo membro ſequentia facile formentur.

Corollarium 2.

377. Deinde ſi ſit $p = n$, ob formulam integrabilem liquet eſſe $(\frac{n}{q}) = (\frac{q}{n}) = \frac{1}{q}$, item $(\frac{p}{n}) = (\frac{n}{p}) = \frac{1}{p}$. Porro cum

$$\int x^{p-1} \partial x (1 - x^n)^{-\frac{p}{n}} = \frac{\pi}{n \sin. \frac{p\pi}{n}},$$

ob $q - n = -p$ ſeu $p + q = n$ erit

$$(\frac{p}{n-p}) = (\frac{n-p}{p}) = \frac{\pi}{n \sin. \frac{p\pi}{n}}.$$

Quare valor formulae $(\frac{p}{q})$ abſolute aſſignari poteſt, quoties fuerit vel $p = n$, vel $q = n$, vel $p + q = n$.

Corollarium 3.

378. Quia etiam inuenimus hanc reductionem

$$\int x^{p+n-1} \partial x (1 - x^n)^{\frac{q-n}{n}} = \frac{p}{p+q} \int x^{p-1} \partial x (1 - x^n)^{\frac{q-n}{n}},$$

ſequitur fore $(\frac{p+n}{q}) = \frac{p}{p+q} \cdot (\frac{p}{q})$: hincque

$$(\frac{p}{q}) = (\frac{q}{p}) = \frac{p-n}{p+q-n} (\frac{p-n}{q}) = \frac{q-n}{p+q-n} (\frac{p}{q-n});$$

tum vero etiam

$$(\frac{p}{q}) = \frac{(p-n)(q-n)}{(p+q-n)(p+q-2n)} \cdot [\frac{p-n}{q-n}]:$$

vnde ſemper numeri p et q infra n deprimi poſſunt.

Problema 46.

379. Inuenire diuerſa producta ex binis huiusmodi formulis, quae inter ſe ſint aequalia.

Solu-

Solutio.

Quaerantur ergo numeri a, b, c, d, et p, q, r, s, vt fiat $(\frac{a}{b})(\frac{c}{d}) = (\frac{p}{q})(\frac{r}{s})$, quod cum fit

$$(\frac{a}{b}) = \frac{a+b}{ab} \cdot \frac{n(a+b+n)}{(a+n)(b+n)} \text{ etc. } (\frac{c}{d}) = \frac{c+d}{cd} \cdot \frac{n(c+d+n)}{(c+n)(d+n)} \text{ etc.}$$

$$(\frac{p}{q}) = \frac{p+q}{pq} \cdot \frac{n(p+q+n)}{(p+n)(q+n)} \text{ etc. } (\frac{r}{s}) = \frac{r+s}{rs} \cdot \frac{n(r+s+n)}{(r+n)(s+n)} \text{ etc.}$$

eueniet, si fuerit

$$\frac{(a+b)(c+d)}{abcd} = \frac{(p+q)(r+s)}{pqrs}, \text{ seu}$$

$$abcd(p+q)(r+s) = pqrs(a+b)(c+d)$$

ita vt, cum vtrinque sex sint factores, singuli singulis sint aequales. Ex quaternis ergo $abcd$ et $pqrs$ binos ad minimum aequales esse oportet: sit itaque $s = d$ efficique oportet

$$abc(p+q)(r+d) = pqr(a+b)(c+d).$$

I. Sumatur alter factor r, qui cum ipsi c aequari nequeat, quia alioquin fieret $(\frac{c}{d}) = (\frac{r}{s})$, statuatur $r = b$, vt fiat

$$ac(p+q)(b+d) = pq(a+b)(c+d).$$

Hic neque p neque q ipsi $p+q$ aequari potest, poni ergo debet.

1.) Vel $p+q = a+b$, vt sit $ac(b+d) = pq(c+d)$, quia neque c neque $(b+d)$ ipsi $c+d$ aequari potest, fieret enim vel $d = 0$, vel $b = c$, et $(\frac{r}{s}) = (\frac{c}{d})$, relinquitur $a = c+d$, et $pq = c(b+d)$; ideoque $p = b+d$ et $q = c$, vnde conficitur:

$$(\frac{c+d}{b})(\frac{c}{d}) = (\frac{b+d}{c})(\frac{b}{d}).$$

2.) Vel $p+q = c+d$, ergo $ac(b+d) = pq(a+b)$, hic c neque ipsi p neque q aequari potest, fieret enim $(\frac{p}{q}) = (\frac{c}{d})$, vnde fiat $c = a+b$, vt sit $pq = a(b+d)$; ergo $p =$

$p=a$; $q=b+d$; $r=b$; $s=d$ consequenter

$$\left(\frac{a}{b}\right)\left(\frac{a+b}{d}\right)=\left(\frac{b+d}{a}\right)\left(\frac{b}{d}\right).$$

II. Quia $r=b$ non differt a praecedenti, ob a et b permutabiles, statuatur $r=p+q$, fietque

$$a\,b\,c\,(d+p+q)=p\,q\,(a+b)\,(c+d).$$

Quoniam r ipsi c aequari nequit, factor $d+p+q$ neque ipsi p, neque q, neque $c+d$ aequalis poni potest, relinquitur ergo $d+p+q=a+b$, et $a\,b\,c=p\,q\,(c+d)$, vbi quia c ipsi $c+d$ aequari nequit, ac p et q pari conditione gaudent, fiat $p=c$; erit $q=a+b-c-d$, et $ab=(c+d)(a+b-c-d)$; vnde $a=c+d$; $q=b$; $p=c$; $r=b+c$; $s=d$; sicque conficitur:

$$\left(\frac{c+d}{b}\right)\left(\frac{c}{d}\right)=\left(\frac{c}{b}\right)\left(\frac{b+c}{d}\right).$$

Corollarium 1.

380. Hae solutiones eodem fere redeunt, indeque tria producta binarum formularum, aequalia eruuntur:

$$\left(\frac{c}{d}\right)\left(\frac{c+d}{b}\right)=\left(\frac{c}{b}\right)\left(\frac{b+c}{d}\right)=\left(\frac{b}{d}\right)\left(\frac{b+d}{c}\right)$$

vel in litteris p, q, r,

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right).$$

Corollarium 2.

381. Si hae formulae in producta infinita euoluantur, reperietur

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\frac{p+q+r}{pqr}\cdot\frac{nn(p+q+r+n)}{(p+n)(q+n)(r+n)}\cdot\frac{4nn(p+q+r+2n)}{(p+2n)(q+2n)(r+2n)}\text{ etc.}$$

vnde patet, tres litteras p, q, r, vtcunque inter se permutari posse, atque hinc ternas illas formulas concludere licet.

Corol-

Corollarium 3.

382. Restituamus ipsas formulas integrales, et sequentia tria producta erunt inter se aequalia

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \cdot \int \frac{x^{p+q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} =$$

$$\int \frac{x^{q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{q+r-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}} =$$

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{p+r-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}.$$

Corollarium 4.

383. Hic casus notatu dignus, quo $p+q=n$, tum enim ob

$$\left(\frac{p+q}{r}\right) = \left(\frac{n}{p}\right) = \frac{1}{r} \text{ et } \left(\frac{p}{q}\right) = \frac{\pi}{n \sin. \frac{p\pi}{n}},$$

haec tria producta fient $= \frac{\pi}{n r \sin. \frac{p\pi}{n}}$. Erit scilicet

$$\int \frac{x^{n-p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{n-p+r-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}} = \int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{p+r-1}\partial x}{\sqrt[n]{(1-x^n)^{p}}}$$

$$= \frac{\pi}{n r \sin. \frac{p\pi}{n}}.$$

Scholion.

384. Triplex ista proprietas productorum ex binis formulis maxime est notatu digna, ac pro variis numeris loco p, q, r substituendis obtinebuntur sequentes aequalitates speciales:

p	q	r	
1	1	2	$\left(\frac{1}{1}\right)\left(\frac{2}{2}\right) = \left(\frac{2}{1}\right)\left(\frac{3}{1}\right)$
1	2	2	$\left(\frac{2}{1}\right)\left(\frac{3}{2}\right) = \left(\frac{2}{2}\right)\left(\frac{4}{1}\right)$
1	2	3	$\left(\frac{2}{1}\right)\left(\frac{3}{3}\right) = \left(\frac{3}{2}\right)\left(\frac{5}{1}\right) = \left(\frac{3}{1}\right)\left(\frac{4}{2}\right)$
1	1	3	$\left(\frac{1}{1}\right)\left(\frac{3}{2}\right) = \left(\frac{3}{1}\right)\left(\frac{4}{1}\right)$
2	2	3	$\left(\frac{2}{2}\right)\left(\frac{4}{3}\right) = \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)$
1	3	3	$\left(\frac{3}{1}\right)\left(\frac{4}{3}\right) = \left(\frac{3}{3}\right)\left(\frac{6}{1}\right)$
2	3	3	$\left(\frac{3}{2}\right)\left(\frac{5}{3}\right) = \left(\frac{3}{3}\right)\left(\frac{6}{2}\right)$
1	1	4	$\left(\frac{1}{1}\right)\left(\frac{4}{2}\right) = \left(\frac{4}{1}\right)\left(\frac{5}{1}\right)$
1	2	4	$\left(\frac{2}{1}\right)\left(\frac{4}{3}\right) = \left(\frac{4}{2}\right)\left(\frac{6}{1}\right) = \left(\frac{4}{1}\right)\left(\frac{5}{2}\right)$
1	3	4	$\left(\frac{3}{1}\right)\left(\frac{4}{4}\right) = \left(\frac{4}{1}\right)\left(\frac{5}{3}\right) = \left(\frac{4}{3}\right)\left(\frac{7}{1}\right)$
1	4	4	$\left(\frac{4}{1}\right)\left(\frac{5}{4}\right) = \left(\frac{4}{4}\right)\left(\frac{8}{1}\right)$
2	2	4	$\left(\frac{2}{2}\right)\left(\frac{4}{4}\right) = \left(\frac{4}{2}\right)\left(\frac{6}{2}\right)$
2	3	4	$\left(\frac{3}{2}\right)\left(\frac{5}{4}\right) = \left(\frac{4}{3}\right)\left(\frac{7}{2}\right) = \left(\frac{4}{2}\right)\left(\frac{6}{3}\right)$
2	4	4	$\left(\frac{4}{2}\right)\left(\frac{6}{4}\right) = \left(\frac{4}{4}\right)\left(\frac{8}{2}\right)$
3	3	4	$\left(\frac{3}{3}\right)\left(\frac{6}{4}\right) = \left(\frac{4}{3}\right)\left(\frac{7}{3}\right)$
3	4	4	$\left(\frac{4}{3}\right)\left(\frac{7}{4}\right) = \left(\frac{4}{4}\right)\left(\frac{8}{3}\right)$.

Quae formulae pro omnibus numeris n valent, ac si numeri maiores quam n occurrant, eos ad minores reduci posse supra vidimus.

Problema 47.

385. Inuenire producta diuersa ex ternis huiusmodi formulis, quae inter se sint aequalia.

Solutio.

Consideretur productum $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)$, quod euolutum praebet:

$$\frac{p+q+r+s}{pqrs} \cdot \frac{n^3(p+q+r+s+n)}{(p+n)(q+n)(r+n)(s+n)} \text{ etc.}$$

quo eundem valorem retinere euidens est, quomodocunque quatuor litterae inter se commutentur. Tum vero eadem euolutio prodit ex hoc producto: $\left(\frac{p}{q}\right)\left(\frac{r}{s}\right)\left(\frac{p+q}{r+s}\right)$, vbi eadem

per-

permutatio locum habet. Aequalia ergo sunt inter se omnia haec producta:

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{p}{s}\right)\left(\frac{p+s}{q}\right)\left(\frac{p+q+s}{r}\right);$$
$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+s}{r}\right);\ \left(\frac{p}{q}\right)\left(\frac{p+r}{s}\right)\left(\frac{p+r+s}{q}\right);\ \left(\frac{p}{s}\right)\left(\frac{p+s}{r}\right)\left(\frac{p+r+s}{q}\right);$$
$$\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{q}{s}\right)\left(\frac{q+s}{p}\right)\left(\frac{p+q+s}{r}\right);\ \left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right);$$
$$\left(\frac{q}{r}\right)\left(\frac{q+r}{s}\right)\left(\frac{q+r+s}{p}\right);\ \left(\frac{q}{s}\right)\left(\frac{q+s}{r}\right)\left(\frac{q+r+s}{p}\right);\ \left(\frac{r}{s}\right)\left(\frac{r+s}{q}\right)\left(\frac{q+r+s}{p}\right).$$

Producta alterius formae ope praecedentis proprietatis hinc sponte fluunt: est enim

$$\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

Deinde vero etiam hoc productum $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)$ euolutum pro primo membro dat: $\frac{(p+q+r)(p+r+s)}{pqrs(p+r)}$, in quo tam p et r, quam q et s inter se permutare licet, ita vt sit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right).$$

Scholion.

386. Quantumuis late haec patere videantur, tamen nullas nouas comparationes suppeditant, quae non iam in praecedenti contineantur. Postrema enim aequalitas

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right)$$

oritur ex multiplicatione harum
$$\left\{\begin{array}{l}\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)\\ \left(\frac{p}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right).\end{array}\right.$$

Priorum vero formatio ex hoc exemplo patebit, aequalitas $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right)$

oritur ex multiplicatione harum
$$\left\{\begin{array}{l}\left(\frac{p}{q}\right)\left(\frac{p+q}{r+s}\right)=\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right)\\ \left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).\end{array}\right.$$

Iſtae autem comparationes praecipue vtiles ſunt ad valores diuerſarum formularum eiusdem ordinis ſeu pro dato numero n inuicem reducendos, vt integratio ad pauciſſimas reuocetur, quibus datis reliquae per eas definiri queant.

Problema 48.

387. Formulas ſimpliciſſimas exhibere, ad quas integratio omnium caſuum in forma $\left(\frac{p}{q}\right) = \int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$ contentorum reduci queat.

Solutio.

Primo eſt $\left(\frac{n}{p}\right) = \frac{1}{p}$, vnde habentur hi caſus

$$\left(\frac{n}{1}\right) = 1;\ \left(\frac{n}{2}\right) = \frac{1}{2};\ \left(\frac{n}{3}\right) = \frac{1}{3};\ \left(\frac{n}{4}\right) = \frac{1}{4};\ \left(\frac{n}{5}\right) = \frac{1}{5} \text{ etc.}$$

Deinde eſt $\left(\frac{p}{n-p}\right) = \frac{\pi}{n \sin. \frac{p\pi}{n}}$ vnde omnium harum formularum valores ſunt cogniti, quas indicemus:

$$\left(\frac{n-1}{1}\right) = \alpha;\ \left(\frac{n-2}{2}\right) = \beta;\ \left(\frac{n-3}{3}\right) = \gamma;\ \left(\frac{n-4}{4}\right) = \delta \text{ etc.}$$

Verum hi non ſufficiunt ad reliquos omnes expediendos, praeterea tanquam cognitos ſpectari oportet hos:

$$\left(\frac{n-2}{1}\right) = A;\ \left(\frac{n-3}{2}\right) = B;\ \left(\frac{n-4}{3}\right) = C;\ \left(\frac{n-5}{4}\right) = D \text{ etc.}$$

atque ex his reliqui omnes determinari poterunt ope aequationum ſupra demonſtratarum; vnde potiſſimum has notaſſe iuuabit:

$$\left(\frac{n-a}{a}\right)\left(\frac{n}{b}\right) = \left(\frac{n-a}{b}\right)\left(\frac{n-a+b}{a}\right);$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-a-b}{b}\right) = \left(\frac{n-b}{b}\right)\left(\frac{n-a-b}{a}\right);$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-b-1}{b}\right)\left(\frac{n-a-b}{a-1}\right) = \left(\frac{n-b}{b}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-b}{a}\right).$$

Ex harum prima poſito $a = b + 1$ inuenitur

n—

$$\left(\frac{n-1}{a}\right) = \left(\frac{n-a}{a}\right)\left(\frac{n}{a-1}\right) : \left(\frac{n-a}{a-1}\right),$$

vbi $\left(\frac{n}{a-1}\right) = \frac{1}{a-1}$, ideoque per formulas assumtas definitur $\left(\frac{n-1}{a}\right)$.

Ex secunda posito $b = 1$ inuenitur

$$\left(\frac{n-a-1}{1}\right) = \left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right) : \left(\frac{n-a}{a}\right).$$

Ex tertia posito $b = 1$ deducitur

$$\left(\frac{n-a-1}{a-1}\right) = \left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right) : \left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$$

sicque reperiuntur omnes formulae $\left(\frac{n-a-2}{a}\right)$, et ex his porro ponendo $b = 2$ in tertia

$$\left(\frac{n-a-2}{a-1}\right) = \left(\frac{n-2}{2}\right)\left(\frac{n-a}{n-1}\right)\left(\frac{n-a-2}{a}\right) : \left(\frac{n-a}{a}\right)\left(\frac{n-3}{2}\right)$$

vnde reperiuntur formae $\left(\frac{n-a-3}{a}\right)$, et ita porro omnes $\left(\frac{n-a-b}{a}\right)$, quippe quae forma omnes complectitur. Labor autem per priores aequationes non mediocriter contrahitur. Inuenta enim $\left(\frac{n-a-2}{a}\right)$ ex prima colligitur

$$\left(\frac{n-2}{a+2}\right) = \left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right) : \left(\frac{n-a-2}{a}\right)$$

ex secunda vero

$$\left(\frac{n-a-2}{2}\right) = \left(\frac{n-2}{2}\right)\left(\frac{n-a-2}{a}\right) : \left(\frac{n-a}{a}\right),$$

similique modo ex inuentis formulis $\left(\frac{n-a-3}{a}\right)$ deriuantur hae

$$\left(\frac{n-3}{a+3}\right) = \left(\frac{n-a-3}{a+3}\right)\left(\frac{n}{a}\right) : \left(\frac{n-a-3}{a}\right)$$

$$\left(\frac{n-a-3}{3}\right) = \left(\frac{n-3}{3}\right)\left(\frac{n-a-3}{a}\right) : \left(\frac{n-a}{a}\right).$$

Corollarium 1.

388. Ex aequatione $\left(\frac{n-1}{a}\right) = \frac{1}{a-1}\left(\frac{n-a}{a}\right) : \left(\frac{n-a}{a-1}\right)$ definiuntur

$$\left(\frac{n-1}{1}\right) = \frac{\beta}{1A};\quad \left(\frac{n-1}{2}\right) = \frac{\gamma}{2B};\quad \left(\frac{n-1}{3}\right) = \frac{\delta}{3C};\quad \left(\frac{n-1}{4}\right) = \frac{\varepsilon}{4D};\ \text{etc.}$$

Ex

Ex aequatione vero $\left(\frac{n-a-1}{1}\right) = \left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right) : \left(\frac{n-a}{a}\right)$ hae formulae

$$\left(\frac{n-2}{1}\right) = \frac{\alpha A}{\alpha};\ \left(\frac{n-3}{1}\right) = \frac{\alpha B}{\beta};\ \left(\frac{n-4}{1}\right) = \frac{\alpha C}{\gamma};\ \left(\frac{n-5}{1}\right) = \frac{\alpha D}{\delta};\ \text{etc.}$$

Corollarium 2.

389. Aequatio

$$\left(\frac{n-a-1}{a-1}\right) = \left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right) : \left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$$

praebet

$$\left(\frac{n-3}{1}\right) = \frac{\alpha AB}{\beta A};\ \left(\frac{n-4}{2}\right) = \frac{\alpha BC}{\gamma A};\ \left(\frac{n-5}{3}\right) = \frac{\alpha CD}{\delta A};\ \left(\frac{n-6}{4}\right) = \frac{\alpha DE}{\varepsilon A}\ \text{etc.}$$

vnde reperiuntur $\left(\frac{n-2}{a+2}\right) = \left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right) : \left(\frac{n-a-2}{a}\right)$ istae formulae

$$\left(\frac{n-2}{3}\right) = \frac{\gamma\beta A}{1\alpha AB};\ \left(\frac{n-2}{4}\right) = \frac{\delta\gamma A}{2\alpha BC};\ \left(\frac{n-2}{5}\right) = \frac{\varepsilon\delta A}{3\alpha CD};\ \left(\frac{n-2}{6}\right) = \frac{\zeta\varepsilon A}{4\alpha DE}\ \text{etc.}$$

atque etiam istae

$$\left(\frac{n-a-2}{a}\right) = \left(\frac{n-2}{2}\right)\left(\frac{n-a-2}{a}\right) : \left(\frac{n-a}{a}\right),\ \text{quae sunt}$$

$$\left(\frac{n-3}{2}\right) = \frac{\beta\alpha AB}{\alpha\beta A};\ \left(\frac{n-4}{2}\right) = \frac{\beta\alpha BC}{\beta\gamma A};\ \left(\frac{n-5}{2}\right) = \frac{\beta\alpha CD}{\gamma\delta A};\ \left(\frac{n-6}{2}\right) = \frac{\beta\alpha DE}{\delta\varepsilon A}\ \text{etc.}$$

Corollarium 3.

390. Tum aequatio

$$\left(\frac{n-a-2}{a-1}\right) = \left(\frac{n-2}{2}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-2}{a}\right) : \left(\frac{n-a}{a}\right)\left(\frac{n-3}{2}\right)\ \text{dat}$$

$$\left(\frac{n-4}{1}\right) = \frac{\alpha\beta ABC}{\beta\gamma AB};\ \left(\frac{n-5}{2}\right) = \frac{\alpha\beta BCD}{\gamma\delta AB};\ \left(\frac{n-6}{3}\right) = \frac{\alpha\beta CDE}{\delta\varepsilon AB};\ \left(\frac{n-7}{4}\right) = \frac{\alpha\beta DEF}{\varepsilon\zeta AB}$$

hinc $\left(\frac{n-3}{a+3}\right) = \left(\frac{n-a-3}{a+3}\right)\left(\frac{n}{a}\right) : \left(\frac{n-a-3}{a}\right)$ praebet

$$\left(\frac{n-3}{4}\right) = \frac{\beta\gamma\delta AB}{1\alpha\beta ABC};\ \left(\frac{n-3}{5}\right) = \frac{\gamma\delta\varepsilon AB}{2\alpha\beta BCD};\ \left(\frac{n-3}{6}\right) = \frac{\delta\varepsilon\zeta AB}{3\alpha\beta CDE}\ \text{etc.}$$

atque ex $\left(\frac{n-a-3}{3}\right) = \left(\frac{n-3}{3}\right)\left(\frac{n-a-3}{a}\right) : \left(\frac{n-a}{a}\right)$ deducuntur

$$\left(\frac{n-5}{3}\right) = \frac{\alpha\beta\gamma BCD}{\beta\gamma\delta AB};\ \left(\frac{n-6}{3}\right) = \frac{\alpha\beta\gamma CDE}{\gamma\delta\varepsilon AB};\ \left(\frac{n-7}{3}\right) = \frac{\alpha\beta\gamma DEF}{\delta\varepsilon\zeta AB}\ \text{etc.}$$

Exemplum 1.

391. *Casus in hac forma* $\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = \left(\frac{p}{q}\right)$ *conten-*

contentos, vbi n $= 2$ *euoluere, vbi est* $(\frac{p+2}{q}) = \frac{p}{p+q}(\frac{p}{q})$.

Manifestum est has formulas omnes vel algebraice vel per angulos expediri, his tamen regulis vtentes, quia numeri p et q binarium superare non debent, vnam formulam a circulo pendentem habemus $(\frac{1}{1}) = \frac{\pi}{2 \sin. \frac{\pi}{2}} = \frac{\pi}{2} = \alpha$, vnde nostri casus erunt:

$(\frac{2}{1}) = 1$; $(\frac{2}{2}) = \frac{1}{2}$

$(\frac{1}{1}) = \alpha$.

Exemplum 2.

392. *Casus in hoc forma* $\int \frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)^{3-q}}} = \left(\frac{p}{q}\right)$ *contentos, vbi* n $= 3$, *euoluere, vbi est* $(\frac{p+3}{q}) = \frac{p}{p+q}(\frac{p}{q})$.

Hic casus principales, ad quos ceteri reducuntur, sunt

$$(\tfrac{2}{1}) = \frac{\pi}{3 \sin. \frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}} = \alpha \text{ et } (\tfrac{1}{1}) = A = \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}},$$

qua concessa erunt reliqui:

$(\frac{3}{1}) = 1$; $(\frac{3}{2}) = \frac{1}{2}$; $(\frac{3}{3}) = \frac{1}{3}$

$(\frac{2}{1}) = \alpha$; $(\frac{2}{2}) = \frac{\alpha}{A}$

$(\frac{1}{1}) = A$.

Exemplum 3.

393. *Casus in hac forma* $\int \frac{x^{p-1}\partial x}{\sqrt[4]{(1-x^4)^{4-q}}} = \left(\frac{p}{q}\right)$ *contentos, vbi* n $= 4$, *euoluere, vbi est* $(\frac{p+4}{q}) = \frac{p}{p+p}(\frac{p}{q})$.

A circulo pendent hae duae

$$\left(\frac{3}{1}\right) = \frac{\pi}{4\,\text{fin.}\,\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}} = \alpha \text{ et } \left(\frac{2}{2}\right) = \frac{\pi}{4\,\text{fin.}\,\frac{2\pi}{4}} = \frac{\pi}{4} = \beta,$$

praeterea vero vna transcendente singulari opus est $\left(\frac{2}{1}\right) = A$, vnde reliquae ita determinantur:

$$\left(\frac{4}{1}\right) = 1;\ \left(\frac{4}{2}\right) = \frac{1}{2};\ \left(\frac{4}{3}\right) = \frac{1}{3};\ \left(\frac{4}{4}\right) = \frac{1}{4}$$
$$\left(\frac{3}{1}\right) = \alpha;\ \left(\frac{3}{2}\right) = \frac{\beta}{A};\ \left(\frac{3}{3}\right) = \frac{\alpha}{2A}$$
$$\left(\frac{2}{1}\right) = A;\ \left(\frac{2}{2}\right) = \beta$$
$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

Exemplum 4.

394. *Casus in hac forma* $\int \frac{x^{p-1}\partial x}{\sqrt[5]{(1-x^5)^{5-q}}} = \left(\frac{p}{q}\right)$ *contentos, vbi* $n = 5$, *euoluere, vbi est* $\left(\frac{p+5}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right)$.

A circulo pendent hae duae formulae:

$$\left(\frac{4}{1}\right) = \frac{\pi}{5\,\text{fin.}\,\frac{\pi}{3}} = \alpha \text{ et } \left(\frac{3}{2}\right) = \frac{\pi}{5\,\text{fin.}\,\frac{2\pi}{5}} = \beta,$$

praeter quas duas nouas transcendentes assumi oportet

$$\left(\frac{3}{1}\right) = A \text{ et } \left(\frac{2}{2}\right) = B,$$

per quas omnes sequenti modo determinantur

$$\left(\frac{5}{1}\right) = 1;\ \left(\frac{5}{2}\right) = \frac{1}{2};\ \left(\frac{5}{3}\right) = \frac{1}{3};\ \left(\frac{5}{4}\right) = \frac{1}{4};\ \left(\frac{5}{3}\right) = \frac{1}{5}$$
$$\left(\frac{4}{1}\right) = \alpha;\ \left(\frac{4}{2}\right) = \frac{\beta}{A};\ \left(\frac{4}{3}\right) = \frac{\beta}{2B};\ \left(\frac{4}{4}\right) = \frac{\alpha}{3A};$$
$$\left(\frac{3}{1}\right) = A;\ \left(\frac{3}{2}\right) = \beta;\ \left(\frac{3}{3}\right) = \frac{\beta\beta}{\alpha B}$$
$$\left(\frac{2}{1}\right) = \frac{\alpha B}{\beta};\ \left(\frac{2}{2}\right) = B$$
$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}$$

Exem-

Exemplum 5.

395. *Casus in hac forma* $\int \frac{x^{p-1}\,\partial x}{\sqrt[6]{(1-x^6)^{6-q}}} = \left(\frac{p}{q}\right)$, *contentos, vbi* $n = 6$, *euoluere.*

A circulo pendent hae tres formulae:

$$\left(\tfrac{5}{1}\right) = \frac{\pi}{6\,\text{fin.}\,\frac{\pi}{6}} = \frac{\pi}{3} = \alpha;\quad \left(\tfrac{4}{2}\right) = \frac{\pi}{6\,\text{fin.}\,\frac{2\pi}{6}} = \frac{\pi}{3\sqrt{3}} = \beta;$$

$$\left(\tfrac{3}{3}\right) = \frac{\pi}{6\,\text{fin.}\,\frac{3\pi}{6}} = \frac{\pi}{6} = \gamma$$

tum vero assumantur hae duae transcendentes:

$$\left(\tfrac{4}{1}\right) = A \text{ et } \left(\tfrac{3}{2}\right) = B$$

atque per has omnes sequenti modo determinantur

$\left(\tfrac{6}{1}\right) = 1$; $\left(\tfrac{6}{2}\right) = \tfrac{1}{2}$; $\left(\tfrac{6}{3}\right) = \tfrac{1}{3}$; $\left(\tfrac{6}{4}\right) = \tfrac{1}{4}$; $\left(\tfrac{6}{5}\right) = \tfrac{1}{5}$; $\left(\tfrac{6}{6}\right) = \tfrac{1}{6}$

$\left(\tfrac{5}{1}\right) = \alpha$; $\left(\tfrac{5}{2}\right) = \tfrac{\beta}{A}$; $\left(\tfrac{5}{3}\right) = \tfrac{\gamma}{2B}$; $\left(\tfrac{5}{4}\right) = \tfrac{\beta}{3B}$; $\left(\tfrac{5}{5}\right) = \tfrac{\alpha}{4A}$

$\left(\tfrac{4}{1}\right) = A$; $\left(\tfrac{4}{2}\right) = \beta$; $\left(\tfrac{4}{3}\right) = \tfrac{\beta\gamma}{\alpha B}$; $\left(\tfrac{4}{4}\right) = \tfrac{\beta\gamma A}{2\alpha BB}$

$\left(\tfrac{3}{1}\right) = \tfrac{\alpha B}{\beta}$; $\left(\tfrac{3}{2}\right) = B$; $\left(\tfrac{3}{3}\right) = \gamma$

$\left(\tfrac{2}{1}\right) = \tfrac{\alpha B}{\gamma}$; $\left(\tfrac{2}{2}\right) = \tfrac{\alpha BB}{\gamma A}$

$\left(\tfrac{1}{1}\right) = \tfrac{\alpha A}{\beta}$.

Scholion.

396. Has determinationes quousque libuerit, continuare licet, in quibus praecipue notari debent casus nouas transcendentium species introducentes; quorum primus occurrit si $n = 3$, estque $\left(\tfrac{1}{1}\right) = \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$, cuius valorem per productum infinitum supra vidimus esse

$$= \tfrac{3}{2} \cdot \tfrac{2.6}{4.4} \cdot \tfrac{5.9}{7.7} \cdot \tfrac{8.12}{10.10} \text{ etc.}$$

quod ex formula $(\frac{1}{1})$, ob $n = 3$, etiam est

$$\frac{2}{1.1} \cdot \frac{3.5}{4.4} \cdot \frac{6.8}{7.7} \cdot \frac{9.11}{10.10} \cdot \frac{12.14}{13.13} \text{ etc.}$$

Deinde ex classe $n = 4$ nascitur haec noua forma transcendens:

$$(\tfrac{2}{1}) = \int \frac{x\,\partial x}{\sqrt[4]{(1-x^4)^3}} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} = \int \frac{\partial x}{\sqrt{(1-x^4)}},$$

quae aequatur huic producto infinito

$$\frac{3}{1.2} \cdot \frac{4.7}{5.6} \cdot \frac{8.11}{9.10} \cdot \frac{12.15}{13.14} \cdot \frac{16.19}{17.18} \text{ etc.} = \frac{3}{2} \cdot \frac{2.7}{5.3} \cdot \frac{4.11}{9.5} \cdot \frac{6.15}{13.7} \cdot \frac{8.19}{17.9} \text{ etc.}$$

Ex classe $n = 5$ impetramus duas nouas formulas transcendentes

$$(\tfrac{3}{1}) = \int \frac{x^2\,\partial x}{\sqrt[5]{(1-x^5)^4}} = \int \frac{\partial x}{\sqrt[5]{(1-x^5)^2}} = \frac{4}{1.3} \cdot \frac{5.9}{6.8} \cdot \frac{10.14}{11.13} \cdot \frac{15.19}{16.18} \text{ etc. et}$$

$$(\tfrac{2}{2}) = \int \frac{x\,\partial x}{\sqrt[5]{(1-x^5)^3}} = \frac{4}{2.2} \cdot \frac{5.9}{7.7} \cdot \frac{10.14}{12.12} \cdot \frac{15.19}{17.17} \text{ etc.}$$

ita vt sit

$$(\tfrac{3}{1}) : (\tfrac{2}{2}) = \frac{2.2}{1.3} \cdot \frac{7.7}{6.8} \cdot \frac{12.12}{11.13} \cdot \frac{17.17}{16.18} \text{ etc.}$$

Classis $n = 6$ has duas formulas transcendentes suppeditat:

$$1.\ (\tfrac{4}{1}) = \int \frac{x^3\,\partial x}{\sqrt[6]{(1-x^6)^5}} = \int \frac{\partial x}{\sqrt[3]{(1-x^6)}} = \tfrac{1}{2} \int \frac{y\,\partial y}{\sqrt[6]{(1-y^3)^5}}$$

$$2.\ (\tfrac{3}{2}) = \int \frac{x^2\,\partial x}{\sqrt[3]{(1-x^6)^2}} = \int \frac{x\,\partial x}{\sqrt{(1-x^6)}} = \tfrac{1}{2} \int \frac{\partial y}{\sqrt{(1-y^3)}} = \tfrac{1}{3} \int \frac{\partial z}{\sqrt[3]{(1-zz)^2}}$$

sumto $y = xx$ et $z = x^3$. Notandum autem est inter has et primum $\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = 2 \int \frac{y\,\partial y}{\sqrt[6]{(1-y^6)^4}} = 2\,(\tfrac{2}{2})$ relationem dari, quae est $2\,\gamma\,(\tfrac{4}{1})(\tfrac{2}{2}) = \alpha\,(\tfrac{3}{2})(\tfrac{3}{2})$, ita vt prima admissa hic altera sufficiat.

CAL-

CALCVLI INTEGRALIS
LIBER PRIOR.

PARS PRIMA,

SEV

METHODVS INVESTIGANDI FVNCTIONES VNIVS VARIABILIS EX DATA RELATIONE QVACVNQVE DIFFERENTIALIVM PRIMI GRADVS.

SECTIO SECVNDA,

DE

INTEGRATIONE AEQVATIONVM DIFFERENTIALIVM.

CAPVT I.

DE SEPARATIONE VARIABILIVM.

Definitio.

§. 397.

In aequatione differentiali *ſeparatio variabilium* locum habere dicitur, cum aequationem ita in duo membra diſpeſcere licet, vt in vtroque vnica tantum variabilis cum ſuo differentiali inſit.

Corollarium 1.

398. Quando igitur aequatio differentialis ita eſt comparata, vt ad hanc formam $X \partial x = Y \partial y$ reduci poſſit, in qua X functio ſit ſolius x et Y ſolius y, tum ea aequatio ſeparationem variabilium admittere dicitur.

Corollarium 2.

399. Quodſi P et X functiones ipſius x tantum, at Q et Y functiones ipſius y tantum denotent, haec aequatio $PY \partial x = QX \partial y$ ſeparationem variabilium admittit, nam per XY diuiſa abit in $\frac{P \partial x}{X} = \frac{Q \partial y}{Y}$, in qua variabiles ſunt ſeparatae.

Corollarium 3.

400. In forma ergo generali $\frac{\partial y}{\partial x} = V$, ſeparatio variabilium locum habet, ſi V eiusmodi fuerit functio ipſarum x et y, vt in duos factores reſolui poſſit, quorum alter ſolam

variabilem x, alter folam y contineat. Si enim fit $V = X Y$, inde prodit aequatio feparata $\frac{\partial y}{Y} = X \partial x$.

Scholion.

401. Pofita differentialium ratione $\frac{\partial y}{\partial x} = p$, in hac fectione eiusmodi relationem inter x, y et p confiderare inftituimus, qua p aequetur functioni cuicunque ipfarum x et y. Hic igitur primum eum cafum contemplamur, quo ifta functio in duos factores refoluitur, quorum alter eft functio tantum ipfius x et alter ipfius y, ita vt aequatio ad hanc formam reduci poffit $X \partial x = Y \partial y$, in qua binae variabiles a fe inuicem feparatae effe dicuntur. Atque in hoc cafu formulae fimplices ante tractatae continentur, quando $Y = 1$, vt fit $\partial y = X \partial x$, et $y = \int X \partial x$, vbi totum negotium ad integrationem formulae $X \partial x$ reuocatur. Haud maiorem autem habet difficultatem aequatio feparata $X \partial x = Y \partial y$, quam perinde ac formulas fimplices tractare licet, id quod in fequente problemate oftendemus.

Problema 49.

402. Aequationem differentialem, in qua variabiles funt feparatae, integrare, feu aequationem inter ipfas variabiles inuenire.

Solutio.

Aequatio feparationem variabilium admittens femper ad hanc formam $Y \partial y = X \partial x$ reducitur; vbi $X \partial x$ tanquam differentiale functionis cuiusdam ipfius x et $Y \partial y$ tanquam differentiale functionis cuiusdam ipfius y fpectari poteft, cum igitur differentialia fint aequalia eorum integralia quoque aequalia effe, vel quantitate conftante differre neceffe eft. Integrentur ergo per praecepta fuperioris fectionis feorfim ambae formulae,

mulae, ſeu quaerantur integralia $\int Y \partial y$ et $\int X \partial x$, quibus inventis erit vtique $\int Y \partial y = \int X \partial x +$ Conſt. qua aequatione ralatio finita inter quantitates x et y exprimetur.

Corollarium 1.

403. Quoties ergo aequatio differentialis ſeparationem variabilium admittit, toties integratio per eadem praecepta, quae ſupra de formulis ſimplicibus ſunt tradita, abſolui poteſt.

Corollarium 2.

404. In aequatione integrali $\int Y \partial y = \int X \partial x +$ Conſt. vel ambae functiones $\int Y \partial y$ et $\int X \partial x$ ſunt algebraicae, vel altera algebraica, altera vero tranſcendens, vel ambae tranſcendentes, ſicque relatio inter x et y vel erit algebraica, vel tranſcendens.

Scholion.

405. In ſeparatione variabilium a nonnullis totum fundamentum reſolutionis aequationum differentialium conſtitui ſolet, ita vt cum aequatio propoſita ſeparationem variabilium non admittit, idonea ſubſtitutio ſit inueſtiganda, cuius beneficio nouae variabiles introductae ſeparationem patiantur. Totum ergo negotium huc reducitur, vt propoſita aequatione differentiali quacunque, eiusmodi ſubſtitutio ſeu nouarum variabilium introductio doceatur, vt deinceps ſeparatio variabilium locum ſit habitura. Optandum vtique eſſet, vt huiusmodi methodus pro quouis caſu idoneam ſubſtitutionem inueniendi aperiretur; ſed nihil omnino certi in hoc negotio eſt compertum, dum pleraeque ſubſtitutiones, quae adhuc in vſu fuerunt, nullis certis principiis innituntur. Deinde autem variabilium ſeparatio non tanquam verum fundamentum omnis integrationis ſpectari poteſt, propterea quod in aequationibus differentialibus ſecundi altiorisue gradus nullum vſum praeſtat; infra

infra autem aliud principium latiſſime patens ſum expoſiturus. In hoc capite interim praecipuas integrationes ope ſeparationis variabilium adminiſtratas exponere operae pretium videtur; quandoquidem in hoc arduo negotio, quam plurimas methodos cognoſcere, plurimum intereſt.

Problema 50.

406. Aequationem differentialem $P\partial x = Q\partial y$, in qua P et Q ſint functiones homogeneae eiusdem dimenſionum numeri ipſarum x et y, ad ſeparationem variabilium reducere, eiusque integrale inuenire.

Solutio.

Cum P et Q ſint functiones homogeneae ipſarum x et y eiusdem dimenſionum numeri, erit $\frac{P}{Q}$ functio homogenea nullius dimenſionis, quae ergo poſito $y = ux$ abit in functionem ipſius u. Ponatur igitur $y = ux$, abeatque $\frac{P}{Q}$ in U functionem ipſius u, ita vt ſit $\partial y = U\partial x$. Sed ob $y = ux$, fit $\partial y = u\partial x + x\partial u$, qua ſubſtitutione noſtra aequatio induet hanc formam $u\partial x + x\partial u = U\partial x$, inter binas variabiles x et u, quae manifeſto ſunt ſeparabiles. Nam diſpoſitis terminis ∂x continentibus ad vnam partem, habetur

$$x\partial u = (U - u)\partial x, \text{ ideoque } \frac{\partial x}{x} = \frac{\partial u}{U - u},$$

quae integrata dat $lx = \int \frac{\partial u}{U - u}$, ita vt iam ex variabili u determinetur x, vnde porro cognoſcitur $y = ux$.

Corollarium 1.

407. Quodſi ergo integrale $\int \frac{\partial u}{U - u}$ etiam per logarithmos exprimi poſſit, ita vt lx aequetur logarithmo functionis cuiuspiam ipſius u, habebitur aequatio algebraica inter x et u, ideoque pro u poſito valore $\frac{y}{x}$, aequatio algebraica inter x et y.

Corol-

Corollarium 2.

408. Cum ſit $y = ux$, erit $ly = lu + lx$, ideoque cum ſit $lx = \int \frac{\partial u}{U - u}$, erit

$$ly = lu + \int \frac{\partial u}{U - u} = \int \frac{\partial u}{u} + \int \frac{\partial u}{U - u};$$

quibus integralibus in vnum reductis, fit $ly = \int \frac{U \partial u}{u(U - u)}$. Verum hic notandum eſt, non in vtraque integratione pro lx et ly conſtantem arbitrariam adiicere licere; ſtatim enim atque alteri integrali eſt adiecta, ſimul conſtans alteri adiicienda definitur, cum eſſe debeat $ly = lx + lu$.

Corollarium 3.

409. Cum ſit

$$\int \frac{\partial u}{U - u} = \int \frac{\partial u - \partial U + \partial U}{U - u} = \int \frac{\partial U}{U - u} - \int \frac{\partial U - \partial u}{U - u},$$

ob hoc poſterius membrum per logarithmos integrabile, erit $lx = \int \frac{\partial U}{U - u} - l(U - u)$, ſeu $lx(U - u) = \int \frac{\partial U}{U - u}$. Perinde ergo eſt, ſiue haec formula $\int \frac{\partial u}{U - u}$ ſiue $\int \frac{\partial U}{U - u}$ integretur.

Scholion.

410. Quoniam haec methodus ad omnes aequationes homogeneas patet, neque etiam ob irrationalitatem, quae forte in functionibus P et Q ineſt, impeditur, imprimis eſt aeſtimanda, plurimumque aliis methodis anteferenda, quae tantum ad aequationes nimis ſpeciales ſunt accommodatae. Atque hinc etiam diſcimus omnes aequationes, quae ope cuiusdam ſubſtitutionis ad homogeneitatem reuocari poſſunt, per eandem methodum tractari poſſe. Veluti ſi proponatur haec aequatio $\partial z + zz\partial x = \frac{a\partial x}{xx}$, ſtatim patet poſito $z = \frac{1}{y}$, eam ad hanc homogeneam $-\frac{\partial y}{yy} + \frac{\partial x}{yy} = \frac{a\partial x}{xx}$, ſeu $xx\partial y = \partial x(xx - ayy)$ reduci. Caeterum non difficulter perſpicitur, vtrum aequatio propoſita huiusmodi ſubſtitutione ad homogeneitatem perduci quaeat?

queat? Plerumque, quoties quidem fieri poteſt, ſufficit has poſitiones $x = u^m$ et $y = v^n$ tentaſſe, vbi facile iudicabitur, num exponentes m et n ita aſſumere liceat, vt vbique idem dimenſionum numerus prodeat, magis enim complicatis ſubſtitutionibus in hoc genere vix locus conceditur, niſi forte quaſi ſponte ſe prodant. Methodum autem integrandi hic expoſitam aliquot exemplis illuſtraſſe iuuabit.

Exemplum 1.

411. *Propoſita aequatione differentiali homogenea* $x\partial x + y\partial y = my\partial x$, *eius integrale inuenire.*

Cum ergo hinc ſit $\frac{\partial y}{\partial x} = \frac{my - x}{y}$, poſito $y = ux$ fit $\frac{my - x}{y} = \frac{mu - 1}{u}$, ideoque ob $\partial y = u\partial x + x\partial u$, erit

$$u\partial x + x\partial u = \frac{(mu-1)}{u}\partial x, \text{ hincque}$$

$$\frac{\partial x}{x} = \frac{u\partial u}{mu - 1 - uu} = \frac{-u\partial u}{1 - mu + uu}, \text{ ſeu}$$

$$\frac{\partial x}{x} = \frac{-u\partial u + \frac{1}{2}m\partial u}{1 - mu + uu} - \frac{\frac{1}{2}m\partial u}{1 - mu + uu};$$

vnde integrando

$$lx = -\tfrac{1}{2}l(1 - mu + uu) - \tfrac{1}{2}m\int\frac{\partial u}{1 - mu + uu} + \text{Conſt.}$$

vbi tres caſus ſunt conſiderandi, prout $m > 2$, vel $m < 2$, vel $m = 2$.

1.) Sit $m > 2$, et $1 - mu + uu$ huiusmodi formam habebit $(u - a)(u - \frac{1}{a})$, vt ſit $m = a + \frac{1}{a} = \frac{aa+1}{a}$, et ob

$$\frac{\partial u}{(u-a)(u-\frac{1}{a})} = \frac{a}{aa-1}\cdot\frac{\partial u}{u-a} - \frac{a}{aa-1}\cdot\frac{\partial u}{u-\frac{1}{a}}, \text{ fiet}$$

$$lx = -\tfrac{1}{2}l(1 - mu + uu) - \frac{(aa+1)}{2(aa-1)}\,l\frac{u-a}{u-\frac{1}{a}} + C, \text{ ſeu}$$

lx

$$l\,x\,\sqrt{(1-m\,u+u\,u)}+\frac{a\,a+1}{2(a\,a-1)}\,l\,\frac{a\,u-a\,a}{a\,u-1}=l\,c,$$

et restituto valore $u=\frac{y}{x}$, aequatio integralis erit

$$l\sqrt{(x\,x-m\,x\,y+y\,y)}+\frac{a\,a+1}{2(a\,a-1)}\,l\,\frac{a\,y-a\,a\,x}{a\,y-x}=l\,c, \text{ seu}$$

$$\left(\frac{a\,y-a\,a\,x}{a\,y-x}\right)^{\frac{a\,a+1}{2(a\,a-1)}}\sqrt{(x\,x-m\,x\,y+y\,y)}=C.$$

2.) Sit $m<2$ seu $m=2\operatorname{cos}.\alpha$, erit
$\int\frac{\partial u}{1-2u\operatorname{cos}.\alpha+u\,u}=\frac{1}{\operatorname{sin}.\alpha}$ Ang. tang. $\frac{u\operatorname{sin}.\alpha}{1-u\operatorname{cos}.\alpha}$:

vnde

$$l\,x\sqrt{(1-m\,u+u\,u)}=C-\frac{\operatorname{cos}.\alpha}{\operatorname{sin}.\alpha}\text{ Ang. tang. }\frac{u\operatorname{sin}.\alpha}{1-u\operatorname{cos}.\alpha},\text{ seu}$$

$$l\sqrt{(x\,x-m\,x\,y+y\,y)}=C-\frac{\operatorname{cos}.\alpha}{\operatorname{sin}.\alpha}\text{ Ang. tang. }\frac{y\operatorname{sin}.\alpha}{x-y\operatorname{cos}.\alpha}.$$

3.) Sit $m=2$, erit $\int\frac{\partial u}{(1-u)^2}=\frac{1}{1-u}$, hincque
$l\,x(1-u)=C-\frac{1}{1-u}$, seu $l(x-y)=B-\frac{x}{x-y}$.

Exemplum 2.

412. *Proposita aequatione differentiali homogenea*

$$\partial x(\alpha x+\beta y)=\partial y(\gamma x+\delta y)$$

eius integrale inuenire.

Posito $y=u\,x$, erit $u\,\partial x+x\,\partial u=\partial x.\frac{\alpha+\beta u}{\gamma+\delta u}$, ideoque

$$\frac{\partial x}{x}=\frac{\partial u(\gamma+\delta u)}{\alpha+\beta u-\gamma u-\delta uu}=\frac{\partial u(\delta u+\frac{1}{2}\gamma-\frac{1}{2}\beta)+\partial u(\frac{1}{2}\gamma+\frac{1}{2}\beta)}{\alpha+(\beta-\gamma)u-\delta u u};$$

vnde integrando

$$l\,x=C-l\sqrt{[\alpha+(\beta-\gamma)u-\delta u u]}+\tfrac{1}{2}(\beta+\gamma)\int\frac{\partial u}{\alpha+(\beta-\gamma)u-\delta u u}:$$

vbi iidem casus, qui ante, sunt considerandi, prout scilicet

 deno-

denominator $\alpha + (\beta - \gamma) u - \delta u u$ vel duos factores habet reales et inaequales, vel aequales, vel imaginarios.

Exemplum 3.

413. *Proposita aequatione differentiali homogenea*

$$x \partial x + y \partial y = x \partial y - y \partial x$$

eius integrale inuenire.

Cum hinc fit $\frac{\partial y}{\partial x} = \frac{x+y}{x-y}$, posito $y = u x$, fit $u \partial x + x \partial u = \frac{1+u}{1-u} \partial x$, seu $x \partial u = \frac{1+uu}{1-u} \partial x$, vnde colligitur $\frac{\partial x}{x} = \frac{\partial u - u \partial u}{1+uu}$, et integrando

$$l x = \text{Ang. tang. } u - l \sqrt{(1 + u u)} + C, \text{ seu}$$

$$l \sqrt{(x x + y y)} = C + \text{Ang. tang. } \frac{y}{x}.$$

Exemplum 4.

414. *Proposita aequatione differentiali homogenea*

$$x x \partial y = (x x - a y y) \partial x$$

eius integrale inuenire.

Hic ergo est $\frac{\partial y}{\partial x} = \frac{xx - ayy}{xx}$, et posito $y = u x$, prodit $u \partial x + x \partial u = (1 - a u u) \partial x$, ideoque $\frac{\partial x}{x} = \frac{\partial u}{1 - u - a u u}$ et $l x = \int \frac{\partial u}{1 - u - a u u}$, cuius euolutioni non opus est immorari.

Exemplum 5.

415. *Proposita aequatione differentiali homogenea*

$$x \partial y - y \partial x = \partial x \sqrt{(x x + y y)}$$

eius integrale inuenire.

Erit ergo $\frac{\partial y}{\partial x} = \frac{y + \sqrt{(xx+yy)}}{x}$, vnde posito $y = u x$, fit $u \partial x + x \partial u = [u + \sqrt{(1 + uu)}] \partial x$, seu $x \partial u = \partial x \sqrt{(1 + u u)}$: ita

ita vt fit $\frac{\partial x}{x} = \frac{\partial u}{\sqrt{(1+uu)}}$, cuius integrale eft

$$l\,x = l\,a + l\,[u + \sqrt{(1+uu)}] = l\,a + l\,(\tfrac{y+\sqrt{(xx+yy)}}{x}),$$

feu $l\,x = l\,a + l\,\frac{x}{\sqrt{(xx+yy)}-y}$, vnde colligitur $x = \frac{ax}{\sqrt{(xx+yy)}-y}$, feu $\sqrt{(xx+yy)} = a + y$, hincque $xx = aa + 2ay$.

Scholion.

416. Huc etiam functiones tranfcendentes numerari poffunt, modo afficiant functiones nullius dimenfionis ipfarum x et y, quia pofito $y = ux$ fimul in functiones ipfius u abeunt. Ita fi in aequatione $P\partial x = Q\partial y$, praeterquam quod P et Q funt functiones homogeneae eiusdem dimenfionum numeri, infint huiusmodi formulae

$$l\frac{\sqrt{(xx+yy)}}{x};\; e^{y:x};\; \text{Ang. fin. } \frac{x}{\sqrt{(xx+yy)}};\; \text{cof. } \frac{nx}{y};\; \text{etc.}$$

methodus expofita pari fucceffu adhiberi poteft, quia pofito $y = ux$ ratio $\frac{\partial y}{\partial x}$, aequatur functioni folius nouae variabilis u.

Problema 51.

417. Aequationem differentialem primi ordinis

$$\partial x(\alpha + \beta x + \gamma y) = \partial y(\delta + \varepsilon x + \zeta y)$$

ad feparationem variabilium reuocare et integrare.

Solutio.

Ponatur $\alpha + \beta x + \gamma y = t$ et $\delta + \varepsilon x + \zeta y = u$, vt fiat $t\,\partial x = u\,\partial y$. At inde colligimus

$$x = \frac{\zeta t - \gamma u + \alpha\zeta + \gamma\delta}{\beta\zeta - \gamma\varepsilon} \text{ et } y = \frac{\beta u - \varepsilon t + \alpha\varepsilon - \beta\delta}{\beta\zeta - \gamma\varepsilon},$$

hincque $\partial x : \partial y = \zeta\partial t - \gamma\partial u : \beta\partial u - \varepsilon\partial t$, vnde nanciscimur hanc aequationem

$$\zeta t\partial t - \gamma t\partial u = \beta u\partial u - \varepsilon u\partial t, \text{ feu}$$

$$\partial t(\zeta t + \varepsilon u) = \partial u(\beta u + \gamma t),$$

 quae

quae cum ſit homogenea et cum exemplo §. 412. conueniat, integratio iam eſt expedita.

Verum tamen caſus exiſtit, quo haec reductio ad homogeneitatem locum non habet, cum fuerit $\beta\zeta - \gamma\varepsilon = 0$, quoniam tum introductio nouarum variabilium t et u tollitur. Hic ergo caſus peculiarem requirit ſolutionem, quae ita inſtituatur; quoniam tum aequatio propoſita eiusmodi formam eſt habitura

$$\alpha\partial x + (\beta x + \gamma y)\partial x = \delta\partial y + n(\beta x + \gamma y)\partial y$$

ponamus $\beta x + \gamma y = z$, erit $\frac{\partial y}{\partial x} = \frac{\alpha + z}{\delta + nz}$. At $\partial y = \frac{\partial z - \beta\partial x}{\gamma}$, ergo $\frac{\partial z - \beta\partial x}{\gamma} = \frac{\alpha + z}{\delta + nz}\partial x$, vbi variabiles manifeſto ſunt ſeparabiles, ſit enim $\partial x = \frac{\partial z(\delta + nz)}{\alpha\gamma + \beta\delta + (\gamma + n\beta)z}$, cuius integratio logarithmos inuoluit, niſi ſit $\gamma + n\beta = 0$, quo caſu algebraice dat $x = \frac{2\partial z + nzz}{2(\alpha\gamma + \beta\delta)} + C$.

Corollarium 1.

418. Aequatio ergo differentialis primi ordinis, vti vocatur, in genere ad homogeneitatem reduci nequit, ſed caſus, quibus $\beta\zeta = \gamma\varepsilon$, inde excipi debent, qui etiam ad aequationem ſeparatam omnino diuerſam deducunt.

Corollarium 2.

419. Si in his caſibus exceptis ſit $n = 0$, ſeu haec propoſita ſit aequatio $\partial y = \partial x(\alpha + \beta x + \gamma y)$, poſito $\beta x + \gamma y = z$, ob $\delta = 1$, haec oritur aequatio $\partial x = \frac{\partial z}{\alpha\gamma + \beta + \gamma z}$, cuius integrale eſt

$$\gamma x = l\frac{\beta + \alpha\gamma + \gamma z}{C} = l\frac{\beta + \alpha\gamma + \beta\gamma x + \gamma\gamma y}{C}, \text{ ſeu}$$

$$\beta + \gamma(\alpha + \beta x + \gamma y) = C e^{\gamma x}.$$

Pro-

Problema 52.

420. Propofita aequatione differentiali huiusmodi:

$$\partial y + P y \partial x = Q d x$$

in qua P et Q fint functiones quaecunque ipfius x, altera autem variabilis y cum fuo differentiali nusquam plus vna habeat dimenfionem, eam ad feparationem variabilium perducere et integrare.

Solutio.

Quaeratur eiusmodi functio ipfius x, quae fit X, vt facta fubftitutione $y = X u$ aequatio prodeat feparabilis: Tum autem oritur

$$\begin{aligned} X \partial u + u \partial X \quad &= Q \partial x \\ + P X u \partial x & \end{aligned}$$

quam aequationem feparationem admittere euidens eft, fi fuerit $\partial X + P X \partial x = 0$, feu $\frac{\partial X}{X} = - P \partial x$, vnde integratio dat $l X = - \int P \partial x$ et $X = e^{-\int P \partial x}$; hac ergo pro X fumta functione, aequatio noftra transformata erit $X \partial u = Q \partial x$, feu $\partial u = \frac{Q \partial x}{X} = e^{\int P \partial x} Q \partial x$, vnde cum P et Q fint functiones datae ipfius x, erit $u = \int e^{\int P \partial x} Q \partial x = \frac{y}{X}$. Quocirca aequationis propofitae integrale eft $y = e^{-\int P \partial x} \int e^{\int P \partial x} Q \partial x$.

Corollarium 1.

421. Refolutio ergo huius aequationis $\partial y + P y \partial x = Q \partial x$ duplicem requirit integrationem alteram formulae $\int P \partial x$, alteram formulae $\int e^{\int P \partial x} Q \partial x$. Sufficit autem in pofteriori conftantem arbitrariam adiecifse, cum valor ipfius y plus vna non recipiat. Etiamfi enim in priori loco $\int P \partial x$ fcribatur $\int P \partial x + C$, formula pro y manet eadem.

Corol-

Corollarium 2.

422. Dum ergo formula $P\partial x$ integratur, sufficit eius integrale particulare sumi, ideoque constanti ingredienti eiusmodi valorem tribui conuenit, vt integralis forma fiat simplicissima.

Scholion.

423. En ergo aliud aequationum genus non minus late patens quam praecedens homogenearum, quod ad separationem variabilium perduci, hocque modo integrari potest. Vnde autem in Analysin maxima vtilitas redundat, cum hic litterae P et Q functiones quascunque ipsius x denotent. Hoc ergo modo manifestum est, tractari posse hanc aequationem $R\partial x + Py\partial x = Q\partial x$, si etiam R functionem quamcunque ipsius x denotet, facta enim diuisione per R forma proposita prodit, modo loco P et Q scribatur $\frac{P}{R}$ et $\frac{Q}{R}$, ita vt integrale futurum sit

$$y = e^{-\int\frac{P\partial x}{R}} \int \frac{e^{\int\frac{P\partial x}{R}} Q\partial x}{R}$$

Ad huius problematis illustrationem quaedam exempla adiiciamus.

Exemplum 1.

424. *Proposita aequatione differentiali*

$$\partial y + y\partial x = a x^n \partial x$$

eius integrale inuenire.

Cum hic sit $P = 1$ et $Q = a x^n$, erit $\int P\partial x = x$, et aequatio integralis fiet

$$y = e^{-x}\int e^x x^n \partial x,$$

quae si n sit numerus integer positiuus, euadet

$y =$

$$y = e^{-x}[e^x(x^n - nx^{n-1} + n(n-1)x^{n-2} - \text{etc.}) + C]$$

qua euoluta prodit

$$y = Ce^{-x} + x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-2)(n-3)x^{n-3} + \text{etc.}$$

vnde pro fimplicioribus valoribus ipfius n,

fi $n=0$, erit $y=Ce^{-x}+1$;

fi $n=1$, erit $y=Ce^{-x}+x-1$;

fi $n=2$, erit $y=Ce^{-x}+x^2-2x+2.1$;

fi $n=3$, erit $y=Ce^{-x}+x^3-3x^2+3.2x-3.2.1$;

etc.

Corollarium 1.

425. Si ergo conftans C fumatur $=0$, habebitur integrale particulare

$$y = x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \text{etc.}$$

quod ergo eft algebraicum, dummodo n fit numerus integer pofitiuus.

Corollarium 2.

426. Si integrale ita determinari debeat, vt pofito $x=0$, valor ipfius y euanefcat, conftans C aequalis fumi debet vltimo termino conftanti figno mutato, vnde id femper erit tranfcendens.

Exemplum 2.

427. *Propofita aequatione differentiali $(1-xx)\partial y + xy\partial x = a\partial x$ eius integrale inuenire.*

Aequatio ifta per $1-xx$ diuifa ad hanc formam reducitur $\partial y + \frac{xy\partial x}{1-xx} = \frac{a\partial x}{1-xx}$, ita vt fit $P = \frac{x}{1-xx}$; $Q = \frac{a}{1-xx}$; hinc $\int P\partial x = -l\sqrt{(1-xx)}$, et $e^{\int P\partial x} = \frac{1}{\sqrt{(1-xx)}}$, ex quo integrale reperitur:

 $y=$

$$y = \sqrt{(1-xx)} \int \frac{a \partial x}{(1-xx)^{\frac{3}{2}}} = \left(\frac{ax}{\sqrt{(1-xx)}} + C\right) \sqrt{(1-xx)};$$

quocirca integrale quaesitum erit

$$y = ax + C\sqrt{(1-xx)}$$

quod si ita determinari debeat, vt posito $x = 0$ euanescat, sumi oportet $C = 0$, eritque $y = ax$.

Exemplum 3.

428. *Proposita aequatione differentiali* $\partial y + \frac{ny\partial x}{\sqrt{(1+xx)}} = a\partial x$, *eius integrale inuenire.*

Cum hic sit $P = \frac{n}{\sqrt{(1+xx)}}$ et $Q = a$, erit
$\int P\partial x = nl[x + \sqrt{(1+xx)}]$ et
$e^{\int P\partial x} = [x + \sqrt{(1+xx)}]^n$, et
$e^{-\int P\partial x} = [\sqrt{(1+xx)} - x]^n$:

vnde integrale quaesitum erit

$$y = [\sqrt{(1+xx)} - x]^n \int a\partial x [x + \sqrt{(1+xx)}]^n,$$

ad quod euoluendum ponatur $x + \sqrt{(1+xx)} = u$, et fiet $x = \frac{uu-1}{2u}$, hinc $\partial x = \frac{\partial u(1+uu)}{2uu}$, ergo

$$\int u^n \partial x = \frac{u^{n-1}}{2(n-1)} + \frac{u^{n+1}}{2(n+1)} + C.$$

Nunc quia $[\sqrt{(1+xx)} - x]^n = u^{-n}$, erit

$$y = Cu^{-n} + \frac{au^{-1}}{2(n-1)} + \frac{au}{2(n+1)} \text{ siue}$$

$$y = C[\sqrt{(1+xx)} - x]^n + \frac{a}{2(n-1)}[\sqrt{(1+xx)} - x] + \frac{a}{2(n+1)}[\sqrt{(1+xx)} + x]$$

quae expressio ad hanc formam reducitur

$$y = C[\sqrt{(1+xx)} - x]^n + \frac{na}{nn-1}\sqrt{(1+xx)} - \frac{ax}{nn-1},$$

si

si integrale ita determinari debeat, vt posito $x = 0$ fiat $y = 0$, sumi oportet $C = -\frac{na}{nn-1}$.

Problema 53.

429. Proposita aequatione differentiali

$$\partial y + P y \partial x = Q y^{n+1} \partial x,$$

vbi P et Q denotent functiones quascunque ipsius x, eam ad separationem variabilium reducere et integrare.

Solutio.

Haec aequatio posito $\frac{1}{y^n} = z$ statim ad formam modo tractatam reducitur, nam ob $\frac{\partial y}{y} = -\frac{\partial z}{nz}$, aequatio nostra per y diuisa, scilicet $\frac{\partial y}{y} + P \partial x = Q y^n \partial x$, statim abit in $-\frac{\partial z}{nz} + P \partial x = \frac{Q \partial x}{z}$, seu $\partial z - n P z \partial x = -n Q \partial x$, cuius integrale est

$$z = -e^{n\int P \partial x} \int e^{-n\int P \partial x} n Q \partial x, \text{ ideoque}$$

$$\frac{1}{y^n} = -n e^{n\int P \partial x} \int e^{-n\int P \partial x} Q \partial x.$$

Tractari autem potest vt praecedens, quaerendo eiusmodi functionem X, vt facta substitutione $y = X u$ prodeat aequatio separabilis: prodit autem

$$X \partial u + u \partial X + P X u \partial x = X^{n+1} u^{n+1} Q \partial x.$$

Fiat ergo $\partial X + P X \partial x = 0$, seu $X = e^{-\int P \partial x}$, eritque

$$\frac{\partial u}{u^{n+1}} = X^n Q \partial x = e^{-n\int P \partial x} Q \partial x,$$

et integrando

$$-\frac{1}{n u^n} = \int e^{-n\int P \partial x} Q \partial x.$$

Iam

Iam quia $u = \frac{y}{X} = e^{\int P \partial x} y$, habebitur vt ante

$$\frac{1}{y^n} = -n\, e^{n\int P \partial x} \int e^{-n\int P \partial x} Q \partial x.$$

Scholion.

430. Hic ergo caſus a praecedente non differre eſt cenſendus, ita vt hic nihil noui ſit praeſtitum. Atque haec duo genera ſunt fere ſola, quae quidem aliquanto latius pateant, in quibus ſeparatio variabilium obtineri queat. Caeteri caſus, qui ope cuiusdam ſubſtitutionis ad variabilium ſeparationem praeparari poſſunt, plerumque ſunt nimis ſpeciales, quam vt inſignis vſus inde expectari poſſit. Interim tamen aliquot caſus prae caeteris memorabiles hic exponamus.

Problema 54.

431. Propoſita hac aequatione differentiali

$$\alpha y \partial x + \beta x \partial y + x^m y^n (\gamma y \partial x + \delta x \partial y) = 0,$$

eam ad ſeparationem variabilium reducere, et integrare.

Solutio.

Tota aequatione per xy diuiſa, nanciſcimur hanc formam

$$\frac{\alpha \partial x}{x} + \frac{\beta \partial y}{y} + x^m y^n \left(\frac{\gamma \partial x}{x} + \frac{\delta \partial y}{y}\right) = 0,$$

vnde ſtatim has ſubſtitutiones $x^\alpha y^\beta = t$ et $x^\gamma y^\delta = u$ inſigni vſu non eſſe cituras, colligimus: inde enim fit

$$\frac{\alpha \partial x}{x} + \frac{\beta \partial y}{y} = \frac{\partial t}{t} \text{ et } \frac{\gamma \partial x}{x} + \frac{\delta \partial y}{y} = \frac{\partial u}{u},$$

hincque aequatio noſtra $\frac{\partial t}{t} + x^m y^n . \frac{\partial u}{u} = 0$. At ex ſubſtitutione ſequitur

$$x^{\alpha\delta - \beta\gamma} = t^\delta u^{-\beta}, \text{ et } y^{\alpha\delta - \beta\gamma} = u^\alpha t^{-\gamma}, \text{ ideoque}$$

$$x = t^{\frac{\delta}{\alpha\delta - \beta\gamma}} u^{\frac{-\beta}{\alpha\delta - \beta\gamma}}, \text{ et } y = t^{\frac{-\gamma}{\alpha\delta - \beta\gamma}} u^{\frac{\alpha}{\alpha\delta - \beta\gamma}};$$

qui-

quibus substitutis fit

$$\frac{\partial t}{t} + t^{\frac{\delta m - \gamma n}{\alpha\delta - \beta\gamma}} u^{\frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma}} \frac{\partial u}{u} = 0, \text{ ideoque}$$

$$t^{\frac{\gamma n - \delta m}{\alpha\delta - \beta\gamma} - 1} \partial t + u^{\frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma} - 1} \partial u = 0,$$

cuius aequationis integrale est

$$\frac{t^{\frac{\gamma n - \delta m}{\alpha\delta - \beta\gamma}}}{\gamma n - \delta m} + \frac{u^{\frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma}}}{\alpha n - \beta m} = C.$$

Vbi tantum superest vt restituantur valores $t = x^\alpha y^\beta$ et $u = x^\gamma y^\delta$. Caeterum notetur, si fuerit vel $\gamma n - \delta m = 0$ vel $\alpha n - \beta m = 0$, loco illorum membrorum vel $l\, t$ vel $l\, u$ scribi debere.

Scholion.

432. Ad aequationem propositam ducit quaestio, qua eiusmodi relatio inter variabiles x et y quaeritur, vt fiat

$$\int y\, \partial x = a\, x\, y + b\, x^{m+1} y^{n+1};$$

ad hanc enim resoluendam differentialia sumi debent, quo prodit

$$y\,\partial x = a\, x\,\partial y + a\, y\,\partial x + b\, x^m y^n [(m+1) y\,\partial x + (n+1) x\,\partial y],$$

qua aequatione cum nostra forma comparata, est

$$\alpha = a - 1,\ \beta = a,\ \gamma = (m+1) b,\ \text{et } \delta = (n+1) b, \text{ ergo}$$

$$\alpha\delta - \beta\gamma = (n - m) a b - (n+1) b$$

$$\alpha n - \beta m = (n - m) a - n, \text{ et } \gamma n - \delta m = (n - m) b,$$

vnde aequatio integralis fit manifesta.

Problema 55.

433. Proposita hac aequatione differentiali

$$y\,\partial y + \partial y\,(a + b\, x + n\, x\, x) = y\,\partial x\,(c + n\, x),$$

eam ad separationem variabilium reducere, et integrare.

Solutio.

Cum hinc fit $\frac{\partial y}{\partial x} = \frac{y(c+nx)}{y+a+bx+nxx}$, tentetur haec substitutio $\frac{y(c+nx)}{y+a+bx+nxx} = u$, seu $y = \frac{u(a+bx+nxx)}{c+nx-u}$, fierique debet $\partial y = u\,\partial x$, seu

$$\frac{\partial y}{y} = \frac{u\,\partial x}{y} = \frac{\partial x(c+nx-u)}{a+bx+nxx}:$$

at ex logarithmis colligitur

$$\frac{\partial y}{y} = \frac{\partial u}{u} + \frac{\partial x(b+2nx)}{a+bx+nxx} - \frac{n\,\partial x+\partial u}{c+nx-u} = \frac{\partial x(c+nx-u)}{a+bx+nxx},$$

quae contrahitur in

$$\frac{\partial u(c+nx) - nu\,\partial x}{u(c+nx-u)} = \frac{\partial x(c-b-nx-u)}{a+bx+nxx}, \text{ seu}$$

$$\frac{\partial u(c+nx)}{u(c+nx-u)} = \frac{\partial x(na+cc-bc+(b-2c)u+uu)}{(c+nx-u)(a+bx+nxx)},$$

quae per $c+nx-u$ multiplicata manifesto est separabilis, proditque

$$\frac{\partial x}{(a+bx+nxx)(c+nx)} = \frac{\partial u}{u[na+cc-bc+(b-2c)u+uu]},$$

cuius ergo integratio per logarithmos et angulos absolui potest. Casu autem hic vix praeuidendo euenit, vt haec substitutio ad votum successerit, neque hoc problema magnopere iuuabit.

Problema 56.

434. Propositam hanc aequationem differentialem

$$(y-x)\,\partial y = \frac{n\,\partial x(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}},$$

ad separationem variabilium reducere, et integrare.

Solutio.

Ob irrationalitatem duplicem vix vllo modo patet, cuiusmodi substitutione vti conueniat. Eiusmodi certe quaeri conuenit, qua eidem signo radicali non ambae variabiles simul implicentur. Ad hunc scopum commoda videtur haec substitutio

tutio $y = \frac{x-u}{1+xu}$, qua fit $y - x = \frac{-u(1+xx)}{1+xu}$, $1+yy = \frac{(1+xx)(1+uu)}{(1+xu)^2}$, et $\partial y = \frac{\partial x(1+uu) - \partial u(1+xx)}{(1+xu)^2}$: atque his valoribus in noſtra aequatione ſubſtitutis, prodit

$$-u\partial x(1+uu) + u\partial u(1+xx) = n\partial x(1+uu)\sqrt{(1+uu)},$$

quae manifeſto ſeparationem variabilium admittit: colligitur ſcilicet

$$\frac{\partial x}{1+xx} = \frac{u\partial u}{(1+uu)(n\sqrt{(1+uu)}+u)},$$

quae aequatio poſito $1+uu = tt$, concinnior redditur

$$\frac{\partial x}{1+xx} = \frac{\partial t}{t[nt+\sqrt{(tt-1)}]},$$

et ope poſitionis $t = \frac{1+ss}{2s}$, ſublata irrationalitate,

$$\frac{\partial x}{1+xx} = -\frac{2\partial s(1-ss)}{(1+ss)[n+1+(n-1)ss]} = \frac{2\partial s}{1+ss} - \frac{2n\partial s}{n+1+(n-1)ss},$$

cuius integratio nulla amplius laborat difficultate.

Scholion.

435. In hoc caſu praecipue ſubſtitutio $y = \frac{x-u}{1+xu}$ notari meretur, qua duplex irrationalitas tollitur: vnde operae pretium erit videre, quid hac ſubſtitutione generaliori praeſtari poſſit $y = \frac{\alpha x+u}{1+\beta xu}$; inde autem fit

$$\alpha - \beta yy = \frac{(\alpha-\beta uu)(1-\alpha\beta xx)}{(1+\beta xu)^2},\; y - \alpha x = \frac{u(1-\alpha\beta xx)}{1+\beta xu}, \text{ et}$$

$$\partial y = \frac{\partial x(\alpha-\beta uu) + \partial u(1-\alpha\beta xx)}{(1+\beta xu)^2};$$

ac iam facile perſpicitur, in cuiusmodi aequationibus haec ſubſtitutio vſum afferre poſſit; eius ſcilicet beneficio haec duplex irrationalitas $\frac{\sqrt{(\alpha-\beta yy)}}{\sqrt{(1-\alpha\beta xx)}}$ reducitur ad hanc ſimplicem $\frac{\sqrt{(\alpha-\beta uu)}}{1+\beta xu}$, quam porro facile rationalem reddere licet. Atque hic fere ſunt caſus, in quibus reductio ad ſeparabilitatem locum inuenit, quibus probe perpenſis, aditus facile patebit ad reliquos caſus, qui quidem etiamnum ſunt tractati; vnicam vero adhuc in-

inueſtigationem apponam circa caſus, quibus haec aequatio $\partial y + yy\,\partial x = ax^m\,\partial x$ ſeparationem variabilium admittit, quandoquidem ad huiusmodi aequationes frequenter peruenitur, atque haec ipſa aequatio olim inter Geometras omni ſtudio eſt agitata.

Problema 57.

436. Pro aequatione $\partial y + yy\,\partial x = a\,x^m\,\partial x$ valores exponentis m definire, quibus eam ad ſeparationem variabilium reduce licet.

Solutio.

Primo haec aequatio ſponte eſt ſeparabilis caſu $m = 0$, tum enim ob $\partial y = \partial x\,(a - yy)$, fit $\partial x = \frac{\partial y}{a - yy}$. Omnis ergo inueſtigatio in hoc verſatur, vt ope ſubſtitutionum alii caſus ad hunc reducantur.

Ponamus $y = \frac{b}{z}$, et fit $-b\,\partial z + bb\,\partial x = ax^m zz\,\partial x$, quae forma vt propoſitae ſimilis euadat, ſtatuatur $x^{m+1} = t$, vt ſit $x^m\,\partial x = \frac{\partial t}{m+1}$, et $\partial x = \frac{t^{\frac{-m}{m+1}}\,\partial t}{m+1}$, eritque

$$b\,\partial z + \frac{azz\,\partial t}{m+1} = \frac{bb}{m+1}\,t^{\frac{-m}{m+1}}\,\partial t,$$

quae ſumto $b = \frac{a}{m+1}$, ad ſimilitudinem propoſitae propius accedit, vt ſit $\partial z + zz\,\partial t = \frac{a}{(m+1)^2}\,t^{\frac{-m}{m+1}}\,\partial t$. Si ergo haec eſſet ſeparabilis, ipſa propoſita iſta ſubſtitutione ſeparabilis fieret et viciſſim; vnde concludimis, ſi aequatio propoſita ſeparationem admittat caſu $m = n$, eam quoque eſſe admiſſuram caſu $m = \frac{-n}{n+1}$. Hinc autem ex caſu $m = 0$ alius non reperitur.

Pona-

Ponamus $y = \frac{1}{x} - \frac{z}{xx}$, vt fit

$$\partial y = -\frac{\partial x}{xx} - \frac{\partial z}{xx} + \frac{2z\partial x}{x^3}, \text{ et}$$

$$yy\partial x = \frac{\partial x}{xx} - \frac{2z\partial x}{x^3} + \frac{zz\partial x}{x^4},$$

vnde prodit

$$-\frac{\partial z}{xx} + \frac{zz\partial x}{x^4} = ax^m\partial x, \text{ seu}$$

$$\partial z - \frac{zz\partial x}{xx} = -ax^{m+2}\partial x:$$

fit nunc $x = \frac{1}{t}$ et fit $\partial z + zz\partial t = at^{-m-4}\partial t$, quae cum propositae sit similis, discimus, si separatio succedat casu $m = n$, etiam succedere casu $m = -n - 4$.

Ex vno ergo casu $m = n$ consequimur duos, scilicet $m = -\frac{n}{n+1}$ et $m = -n - 4$. Cum igitur constet casus $m = 0$, hinc formulae alternatim adhibitae praebent sequentes

$$m = -4;\ m = -\tfrac{4}{3};\ m = -\tfrac{8}{3};\ m = -\tfrac{8}{5};$$
$$m = -\tfrac{12}{5};\ m = -\tfrac{12}{7};\ m = -\tfrac{16}{7};\ \text{etc.}$$

qui casus omnes in hac formula $m = \frac{-4i}{2i \pm 1}$ continentur.

Corollarium 1.

437. Quodsi ergo fuerit vel $m = \frac{-4i}{2i+1}$, vel $m = \frac{-4i}{2i-1}$, aequatio $\partial y + yy\partial x = ax^m\partial x$ per aliquot substitutiones repetitas tandem ad formam $\partial u + uu\partial v = c\partial v$, cuius separatio et integratio constat, reduci potest.

Corollarium 2.

438. Scilicet si fuerit $m = \frac{-4i}{2i+1}$, aequatio

$$\partial y + yy\partial x = ax^m\partial x$$

per substitutiones $x = t^{\frac{1}{m+1}}$ et $y = \frac{a}{(m+1)z}$ reducitur ad hanc

$\partial z + zz\partial t = \frac{a}{(m+1)^2} t^n \partial t$, vbi $n = \frac{-4i}{2i-1}$, qui casus vno gradu inferior est censendus.

Corollarium 3.

439. Sin autem fuerit $m = \frac{-4i}{2i-1}$, aequatio

$$\partial y + yy\partial x = a x^m \partial x$$

per has substitutiones $x = \frac{1}{t}$ et $y = \frac{1}{x} - \frac{z}{xx}$ seu $y = t - ttz$, reducitur ad hanc $\partial z + zz\partial t = a t^n \partial t$, in qua est

$$n = \frac{-4(i-1)}{2i-1} = \frac{-4(i-1)}{2(i-1)+1},$$

qui casus denuo vno gradu inferior est.

Corollarium 4.

440. Omnes ergo casus separabiles hoc modo inuenti, pro exponente m dant numeros negatiuos intra limites 0 et -4 contentos, ac si i sit numerus infinitus, prodit casus $m = -2$, qui autem per se constat, cum aequatio $\partial y + yy\partial x = \frac{a\partial x}{xx}$, posito $y = \frac{1}{z}$, fiat homogenea.

Scholion 1.

441. Aequatio haec $\partial y + yy\partial x = a x^m \partial x$ vocari solet Riccatiana ab Auctore Comite *Riccati*, qui primus casus separabiles proposuit. Hic quidem eam in forma simplicissima exhibui, cum eo haec $\partial y + Ayy t^\mu \partial t = B t^\lambda \partial t$, ponendo $A t^\mu \partial t = \partial x$ et $A t^{\mu+1} = (\mu+1)x$, statim reducatur. Caeterum etsi binae substitutiones, quibus hic sum vsus, sunt simplicissimae, tamen magis compositis adhibendis nulli alii casus separabiles deteguntur: ex quo hoc omnino memorabile est visum, hanc aequationem rarissime separationem admittere, tametsi numerus casuum, quibus hoc praestari queat, reuera sit infinitus. Caeterum haec inuestigatio ab exponente ad simplicem

cem coefficientem traduci poteſt; poſito enim $y = x^{\frac{m}{2}} z$, prodit $\partial z + \frac{m z \partial x}{2 x} + x^{\frac{m}{2}} z z \partial x = a x^{\frac{m}{2}} \partial x$, vbi ſit fiat $x^{\frac{m}{2}} \partial x = \partial t$, et $x^{\frac{m+2}{2}} = \frac{m+2}{2} t$, erit $\frac{\partial x}{x} = \frac{2 \partial t}{(m+2) t}$, hincque

$$\partial z + \frac{m z \partial t}{(m+2) t} + z z \partial t = a \partial t,$$

quae ergo aequatio, quoties fuerit $\frac{m}{m+2} = \pm 2 i$, ſeu numerus par tam poſitiuus, quam negatiuus, ſeparabilis reddi poteſt, ita vt haec aequatio

$$\partial z \pm \frac{2 i z \partial t}{t} + z z \partial t = a \partial t$$

ſemper ſit integrabilis. Si praeterea ponatur $z = u - \frac{m}{2(m+2) t}$, oritur

$$\partial u + u u \partial t = a \partial t - \frac{m(m+4) \partial t}{4 (m+2)^2 t t},$$

et pro caſibus ſeparabilitatis $m = \frac{-4 i}{2 i \pm 1}$, habebitur

$$\partial u + u u \partial t = a \partial t + \frac{i (i \pm 1) \partial t}{t t}.$$

Vberiorem autem huius aequationis euolutionem, quandoquidem eſt maximi momenti, in ſequentibus docebo; vbi integratione aequationum differentialium per ſeries infinitas ſum acturus, hinc enim facilius caſus ſeparabiles eruemus, ſimulque integralia aſſignare poterimus.

Scholion 2.

442. Ampliora praecepta circa ſeparationem variabilium, quae quidem vſum ſint habitura, vix tradi poſſe videntur, vnde intelligitur in pauciſſimis aequationibus differentialibus hanc methodum adhiberi poſſe. Progrediar igitur ad aliud principium explicandum, vnde integrationes haurire liceat, quod multo latius patet, dum etiam ad aequationes differentiales altiorum graduum accommodari poteſt, ita vt in eo ve-

rus ac naturalis fons omnium integrationum contineri videatur. Iſtud autem principium in hoc conſiſtit, quod propoſita quacunque aequatione differentiali inter duas variabiles, ſemper detur functio quaedam, per quam aequatio multiplicata fiat integrabilis. Aequationis ſcilicet omnia membra ad eandem partem diſponi oportet, vt talem formam obtineat $P\partial x + Q\partial y = o$; ac tum dico ſemper dari functionem quandam variabilium x et y, puta V, vt facta multiplicatione, formula $VP\partial x + VQ\partial y$ integrabilis exiſtat, ſeu vt verum ſit differentiale ex differentiatione cuiuspiam functionis binarum variabilium x et y natum. Quodſi enim haec functio ponatur $= S$, vt ſit $\partial S = VP\partial x + VQ\partial y$, quia eſt $P\partial x + Q\partial y = o$, erit etiam $\partial S = o$, ideoque $S = \text{Conſt.}$ quae ergo aequatio erit integrale idque completum aequationis differentialis $P\partial x + Q\partial y = o$. Totum ergo negotium ad inuentionem illius multiplicatoris V redit.

CAPVT

CAPVT II.

DE INTEGRATIONE AEQVATIONVM OPE MVLTIPLICATORVM.

Problema 58.

443.

Propofitam aequationem differentialem examinare vtrum per fe fit integrabilis nec ne?

Solutio.

Difpofitis omnibus aequationis terminis ad eandem partem figni aequalitatis, vt huiusmodi habeatur forma $P\partial x + Q\partial y = 0$, aequatio per fe erit integrabilis, fi formula $P\partial x + Q\partial y$ fuerit verum differentiale functionis cuiuspiam binarum variabilium x et y. Hoc autem euenit, vti in calculo differentiali oftendimus, fi differentiale ipfius P, fumta fola y variabili, ad ∂y eandem habeat rationem, ac differentiale ipfius Q, fumta fola x variabili, ad ∂x: feu adhibito fignandi modo, quo in Calculo differentiali fumus vfi, fi fuerit $(\frac{\partial P}{\partial y}) = (\frac{\partial Q}{\partial x})$. Nam fi Z fit ea functio, cuius differentiale eft $P\partial x + Q\partial y$, erit hoc fignandi modo $P = (\frac{\partial Z}{\partial x})$ et $Q = (\frac{\partial Z}{\partial y})$: hinc ergo fequitur $(\frac{\partial P}{\partial y}) = (\frac{\partial\partial Z}{\partial x\partial y})$ et $(\frac{\partial Q}{\partial x}) = (\frac{\partial\partial Z}{\partial y\partial x})$. At eft $(\frac{\partial\partial Z}{\partial x\partial y}) = (\frac{\partial\partial Z}{\partial y\partial x})$, vnde colligitur $(\frac{\partial P}{\partial y}) = (\frac{\partial Q}{\partial x})$. Quare propofita aequatione differentiali $P\partial x + Q\partial y = 0$, vtrum ea per fe fit integrabilis nec ne? hoc modo dignofcetur: Quaerantur per differentiationem valores $(\frac{\partial P}{\partial y})$ et $(\frac{\partial Q}{\partial x})$, qui fi

 fuerint

fuerint inter se aequales, aequatio per se erit integrabilis; sin autem hi valores sint inaequales, aequatio non erit per se integrabilis.

Corollarium 1.

444. Omnes ergo aequationes differentiales, in quibus variabiles sunt a se inuicem separatae, per se sunt integrabiles: habebunt enim huiusmodi formam $X\partial x + Y\partial y = 0$, vt X sit functio solius x et Y solius y, eritque propterea

$$\left(\frac{\partial X}{\partial y}\right) = 0 \text{ et } \left(\frac{\partial Y}{\partial x}\right) = 0.$$

Corollarium 2.

445. Vicissim igitur, si proposita aequatione differentiali $P\partial x + Q\partial y = 0$, fuerit $\left(\frac{\partial P}{\partial y}\right) = 0$ et $\left(\frac{\partial Q}{\partial x}\right) = 0$, variabiles in ea erunt separatae; littera enim P erit functio tantum ipsius x et Q tantum ipsius y. Vnde aequationes separatae quasi primum genus aequationum per se integrabilium constituunt.

Corollarium 3.

446. Euidens autem est, fieri posse, vt sit $\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right)$, etiamsi neuter horum valorum sit nihilo aequalis. Dantur ergo aequationes per se integrabiles, licet variabiles in iis non sint separatae.

Scholion.

447. Criterium hoc, quo aequationes per se integrabiles agnoscimus, maximi est momenti in hac, quam tradere suscipimus, methodo integrandi. Quodsi enim aequatio deprehendatur per se integrabilis, eius integrale per praecepta iam exposita inueniri potest; sin autem aequatio non fuerit per se integrabilis, semper dabitur quantitas, per quam si ea multiplicetur, fiat per se integrabilis; vnde totum negotium

eo reuocabitur, vt propofita aequatione quacunque per fe non integrabili, inueniatur multiplicator idoneus, qui eam reddat per fe integrabilem; qui fi femper inueniri poffet, nihil amplius in hac methodo integrandi effet defiderandum. Verum haec inueftigatio rariffime fuccedit, ac vix adhuc latius patet, quam ad eas aequationes, quas ope feparationis variabilium iam tractare docuimus; interim tamen non dubito hanc methodum praecedenti longe praeferre, cum ad naturam aequationum magis videatur accommodata, atque etiam ad aequationes differentiales altiorum graduum pateat, in quibus feparatio variabilium nullius eft vfus.

Problema 59.

448. Aequationis differentialis, quam per fe integrabilem effe conftat, integrale inuenire.

Solutio.

Sit aequatio differentialis $P \partial x + Q \partial y = 0$, in qua cum fit $\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right)$, erit $P \partial x + Q \partial y$ differentiale cuiuspiam functionis binarum variabilium x et y, quae fit Z, vt fit $\partial Z = P \partial x + Q \partial y$. Cum ergo habebimus hanc aequationem $\partial Z = 0$, erit integrale quaefitum $Z = C$. Totum negotium ergo huc redit, vt ifta functio Z eruatur, quod cum fciamus effe $\partial Z = P \partial x + Q \partial y$ haud difficulter praeftabitur. Nam quia fumta tantum x variabili, et altera y vt conftante fpectata, eft $\partial Z = P \partial x$, habemus hic formulam differentialem fimplicem vnicam variabilem x inuoluentem, quae per praecepta fuperioris fectionis integrata dabit $Z = \int P \partial x +$ Conft. vbi autem notandum eft, in hac conftante quantitatem hic pro conftanti habitam y vtcunque ineffe poffe; vnde eius loco fcribatur Y, vt fit $Z = \int P \partial x + Y$. Deinde fimili modo x pro conftante habeatur, fpectata fola y vt variabili, et cum fit

fit $\partial Z = Q \partial y$, erit quoque $Z = \int Q \partial y +$ Conft. quae conftans autem quantitatem x inuoluet, ita vt fit functio ipfius x, qua pofita X, erit $Z = \int Q \partial y + X$. Quanquam autem neque hic functio X neque ibi functio Y determinatur, tamen quia effe debet $\int P \partial x + Y = \int Q \partial y + X$, hinc vtraque determinabitur. Cum enim fit $\int P \partial x - \int Q \partial y = X - Y$, haec quantitas $\int P \partial x - \int Q \partial y$ femper in eiusmodi binas partes diftinguetur, quarum altera eft functio ipfius x tantum, et altera ipfius y tantum, vnde valores X et Y fponte cognofcuntur.

Corollarium 1.

449. Cum fit $Q = (\frac{\partial Z}{\partial y})$, duplici integratione ne opus quidem eft. Inuento enim integrali $\int P \partial x$, id iterum differentietur, fumta fola y variabili, prodeatque $V \partial y$, vnde neceffe eft fiat $V \partial y + \partial Y = Q \partial y$, ideoque

$$\partial Y = Q \partial y - V \partial y = (Q - V) \partial y.$$

Corollarium 2.

450. Aequationum ergo per fe integrabilium $P \partial x + Q \partial y = 0$ integratio ita perficietur. Quaeratur integrale $\int P \partial x$ fpectata y conftante, idque rurfus differentietur fpectata fola y variabili, vnde prodeat $V \partial y$: tum $Q - V$ erit functio ipfius y tantum; vnde quaeratur $Y = \int (Q - V) \partial y$, eritque aequatio integralis $\int P \partial x + Y =$ Conft.

Corollarium 3.

451. Vel quaeratur $\int Q \partial y$ fpectata x conftante, quod integrale rurfus differentietur fumta x variabili, y autem conftante, vnde prodeat $U \partial x$: tum certe erit $P - U$ functio ipfius x tantum; vnde quaeratur $X = \int (P - U) \partial x$, eritque aequatio integralis quaefita $\int Q \partial y + X =$ Conft.

Corol-

Corollarium 4.

452. Ex rei natura patet, perinde esse vtra via procedatur, necesse enim est ad eandem aequationem integralem perueniri, si quidem aequatio differentialis proposita per se fuerit integrabilis. Tum autem certe eueniet, vt priori casu $Q - V$ sit functio solius y, posteriori autem $P - U$ functio solius x.

Scholion.

453. Haec methodus integrandi etiam tentari posset, antequam exploratum esset, num aequatio integrabilis existat; si enim vel in modo Corollarii 2. eueniret, vt $Q - V$ esset functio ipsius y tantum, vel in modo Corollarii 3. vt $P - U$ esset functio ipsius x tantum, hoc ipsum indicio foret, aequationem esse per se integrabilem. Verum tamen praestat ante omnia scrutari, an aequatio integrabilis sit per se nec ne; seu an sit $(\frac{\partial P}{\partial y}) = (\frac{\partial Q}{\partial x})$? quoniam hoc examen sola differentiatione absoluitur. Exempla igitur aliquot aequationum per se integrabilium afferamus, quo non solum methodus integrandi, sed etiam insignes illae proprietates, quas commemorauimus, clarius intelligantur.

Exemplum 1.

454. *Aequationem per se integrabilem*

$$\partial x(\alpha x + \beta y + \gamma) + \partial y(\beta x + \delta y + \varepsilon) = 0,$$

integrare.

Cum hic sit

$$P = \alpha x + \beta y + \gamma \text{ et } Q = \beta x + \delta y + \varepsilon, \text{ erit}$$
$$(\tfrac{\partial P}{\partial y}) = \beta \text{ et } (\tfrac{\partial Q}{\partial y}) = \beta,$$

qua aequalitate integrabilitas per se confirmatur. Quaeratur

ergo per Corollarium 2, spectata y vt constante,

$$\int P\,\partial x = \tfrac{1}{2}\alpha xx + \beta yx + \gamma x, \text{ erit}$$

$$V\,\partial y = \beta x\,\partial y, \text{ et } (Q - V)\,\partial y = \partial y\,(\delta y + \varepsilon) = \partial Y,$$

ideoque $Y = \frac{1}{2}\delta yy + \varepsilon y$, vnde integrale erit

$$\tfrac{1}{2}\alpha xx + \beta yx + \gamma x + \tfrac{1}{2}\delta yy + \varepsilon y = C.$$

Modo autem Corrollarii 3. spectata x constante, erit

$$\int Q\,\partial y = \beta xy + \tfrac{1}{2}\delta yy + \varepsilon y,$$

quae, spectata y constante, praebet $U\,\partial x = \beta y\,\partial x$, hincque

$$(P - U)\,\partial x = (\alpha x + \gamma)\,\partial x, \text{ et } X = \tfrac{1}{2}\alpha xx + \gamma x,$$

vnde $\int Q\,\partial y + X = C$ integrale dat vt ante. Hinc simul etiam intelligitur esse

$$\int P\,\partial x - \int Q\,\partial y = \tfrac{1}{2}\alpha xx + \gamma x - \tfrac{1}{2}\delta yy - \varepsilon y,$$

quae in duas functiones $X - Y$ sponte dispescitur.

Exemplum 2.

455. *Aequationem per se integrabilem*

$$\frac{\partial y}{y} = \frac{x\,\partial y - y\,\partial x}{y\sqrt{(xx+yy)}}, \textit{ seu } \frac{\partial x}{\sqrt{(xx+yy)}} + \frac{\partial y}{y}\left(1 - \frac{x}{\sqrt{(xx+yy)}}\right) = 0$$

integrare.

Cum hic sit

$$P = \frac{1}{\sqrt{(xx+yy)}} \text{ et } Q = \frac{1}{y} - \frac{x}{y\sqrt{(xx+yy)}},$$

pro charactere integrabilitatis per se cognoscendo est

$$\left(\frac{\partial P}{\partial y}\right) = \frac{-y}{(xx+yy)^{\frac{3}{2}}} \text{ et } \left(\frac{\partial Q}{\partial x}\right) = \frac{-y}{(xx+yy)^{\frac{3}{2}}},$$

qui bini valores vtique sunt aequales. Iam pro integrali inueniendo, vtamur regula Corollarii 2. et habebimus

$$\int P\,\partial x = l[x + \sqrt{(xx+yy)}] \text{ et } V\,\partial y = \frac{y\,\partial y}{[x+\sqrt{(xx+yy)}]\sqrt{(xx+yy)}},$$

seu

feu fupra et infra per $\sqrt{(xx+yy)}-x$ multiplicando,

$$V=\frac{\sqrt{(xx+yy)}-x}{y\sqrt{(xx+yy)}}=\frac{1}{y}-\frac{x}{y\sqrt{(xx+yy)}},$$

vnde $Q-V=0$, et $Y=\int(Q-V)\partial y=0$, ficque integrale quaefitum $l[x+\sqrt{(xx+yy)}]=$ Conft.

Per regulam Corollarii 3. habemus

$$\int Q\partial y=ly-x\int\frac{\partial y}{y\sqrt{(xx+yy)}},$$

at pofito $y=\frac{1}{z}$, eft

$$\int\frac{\partial y}{y\sqrt{(xx+yy)}}=-\int\frac{\partial z}{\sqrt{(xxzz+1)}}=-\frac{1}{x}l[xz+\sqrt{(xxzz+1)}],$$

ergo

$$\int Q\partial y=ly+l\frac{x+\sqrt{(xx+yy)}}{y}=l[x+\sqrt{(xx+yy)}],$$

vnde $U\partial x=\frac{\partial x}{\sqrt{(xx+yy)}}$; hinc $(P-U)\partial x=0$.

Exemplum 3.

456. *Aequationem per fe integrabilem*

$$(xx+yy-aa)\partial y+(aa+2xy+xx)\partial x=0,$$

integrare.

Hic ergo eft

$$P=aa+2xy+xx,\text{ et }Q=xx+yy-aa,$$

vnde $\left(\frac{\partial P}{\partial y}\right)=2x$ et $\left(\frac{\partial Q}{\partial x}\right)=2x$, quae aequalitas integrabilitatem per fe innuit. Tum vero eft

$$\int P\partial x=aax+xxy+\tfrac{1}{3}x^3\text{ et }V\partial y=xx\partial y,$$

vnde $(Q-V)\partial y=(yy-aa)\partial y$ et $Y=\frac{1}{3}y^3-aay$.
Ergo integrale

$$aax+xxy+\tfrac{1}{3}x^3+\tfrac{1}{3}y^3-aay=\text{Conft.}$$

Altero modo eft

$$\int Q\partial y=xxy+\tfrac{1}{3}y^3-aay,\text{ hincque}$$

$$U\partial x=2xy\partial x,\text{ ergo}$$

 (P—

$$(P - U)\,\partial x = (aa + xx)\,\partial x \text{ et } X = aax + \tfrac{1}{3}x^3,$$

vnde integrale oritur vt ante.

Scholion.

457. In his exemplis licuit, integrale $\int P\,\partial x$ actu exhibere, indeque eius differentiale $V\,\partial y$ sumta sola y variabili assignare. Quodsi autem hoc integrale $\int P\,\partial x$ euolui nequeat, haud liquet quomodo inde differentiale $V\partial y$ elici possit, quandoquidem formula $\int P\,\partial x$ in se spectata constantem quamcunque, quae etiam y in se implicet, complectitur. Tum igitur quomodo procedendum sit, videamus. Ponamus $Z = \int P\,\partial x + Y$, et cum quaeratur $\left(\frac{\partial.\int P\,\partial x}{\partial y}\right) = V$, ob $\int P\,\partial x = Z - Y$, erit $V = \left(\frac{\partial Z}{\partial y}\right) - \frac{\partial Y}{\partial y}$. At est $\left(\frac{\partial Z}{\partial x}\right) = P$, ergo $\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right) = \left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial V}{\partial x}\right)$, ob $\left(\frac{\partial Z}{\partial y}\right) = V + \frac{\partial Y}{\partial y}$. Hinc erit $V = \int\partial x\left(\frac{\partial P}{\partial y}\right)$, quare quantitas V inuenitur per integrationem huius formulae $\int\partial x\left(\frac{\partial P}{\partial y}\right)$, in qua y vt constans spectatur, postquam in valore $\left(\frac{\partial P}{\partial y}\right)$ inueniendo sola y variabilis esset assumta. Verum cum hic denuo constans cum y implicetur, hinc illa functio Y quam quaerimus non determinatur. Ratio huius incommodi manifesto in ambiguitate integralium $\int P\,\partial x$ et $\int\partial x\left(\frac{\partial P}{\partial y}\right)$ est sita, dum vtraque functiones arbitrarias ipsius y recipit. Remedium ergo afferetur, si vtrumque integrale certa quadam conditione determinetur. Ita quando integrale $\int P\,\partial x$ ita accipi ponimus, vt euanescat posito $x = f$, vbi quidem constantem f pro lubitu accipere licet, tum eadem lege alterum integrale $\int\partial x\left(\frac{\partial P}{\partial y}\right)$ capiatur. Quo facto erit $Q - \int\partial x\left(\frac{\partial P}{\partial y}\right)$ functio ipsius y tantum, et aequationis $P\,\partial x + Q\,\partial y = 0$, integrale erit

$$\int P\,\partial x + \int\partial y\left[Q - \int\partial x\left(\frac{\partial P}{\partial y}\right)\right] = \text{Const.}$$

dum-

dummodo ambo integralia $\int P\,\partial x$ et $\int \partial x\,(\frac{\partial P}{\partial y})$, in quibus y vt constans tractatur, ita determinentur, vt euanescant, dum in vtraque ipsi x idem valor f tribuitur. Quare hinc istam colligimus regulam

Regula pro integratione aequationis per se integrabilis
$P\,\partial x + Q\,\partial y = 0$, in qua $(\frac{\partial P}{\partial y}) = (\frac{\partial Q}{\partial x})$.

458. Quaerantur integralia $\int P\,\partial x$ et $\int \partial x\,(\frac{\partial P}{\partial y})$, spectando y vt constantem, ita vt ambo euanescant, dum ipsi x certus quidam valor, puta $x = f$, tribuitur. Tum erit $Q - \int \partial x\,(\frac{\partial P}{\partial y})$ functio ipsius y tantum, quae sit $= Y$, et integrale quaesitum erit $\int P\,\partial x + \int Y\,\partial y = Const.$

Vel quod eodem redit, quaerantur integralia $\int Q\,\partial y$ et $\int \partial y\,(\frac{\partial Q}{\partial x})$, spectando x vt constantem, ita vt ambo euanescant, dum ipsi y certus quidem valor, puta $y = g$, tribuitur: tum $P = \int \partial y\,(\frac{\partial Q}{\partial x})$ erit functio ipsius x tantum, qua posita $= X$, erit integrale quaesitum $\int Q\,\partial y + \int X\,\partial x = Const.$

Demonstratio.

Veritatem huius regulae ex praecedentibus perspicere licet, si cui forte precario assumsisse videamur, ambas formulas $\int P\,\partial x$ et $\int \partial x\,(\frac{\partial P}{\partial y})$ eadem lege determinari debere, vt dum ipsi x certus quidam valor puta $x = f$ tribuitur, ambae euanescant. Sed ne forte quis putet, alteram integrationem pari iure secundum aliam legem determinari posse, hanc demonstrationem addo. Prima quidem integratio ab arbitrio nostro pendet, quam ergo ita determinari assumamus, vt integrale $\int P\,\partial x$ euanescat posito $x = f$, quo facto dico, alterum integrale $\int \partial x\,(\frac{\partial P}{\partial y})$ necessario per eandem conditionem determinari oportere.

tere. Sit enim $\int P \partial x = Z$, eritque Z eiusmodi functio ipsarum x et y, quae euanescit posito $x = f$; habebit ergo factorem $f - x$, vel eius quampiam potestatem positiuam $(f - x)^\lambda$, ita vt sit $Z = (f - x)^\lambda T$. Nunc quia $\int \partial x \left(\frac{\partial P}{\partial y}\right)$ exprimit valorem ipsius $\left(\frac{\partial Z}{\partial y}\right)$, erit $\int \partial x \left(\frac{\partial P}{\partial y}\right) = (f - x)^\lambda \left(\frac{\partial T}{\partial y}\right)$, ex quo manifestum est hoc integrale etiam euanescere posito $x = f$, ita vt huius integralis determinatio non amplius arbitrio nostro relinquatur. Hoc posito erit vtique aequationis per se integrabilis $P \partial x + Q \partial y = 0$ integrale $\int P \partial x + \int Y \partial y = \text{Const.}$ existente $Y = Q - \int \partial x \left(\frac{\partial P}{\partial y}\right)$; nam posito $\int P \partial x = Z$, quatenus scilicet in hac integratione y pro constante habetur, vt habetur haec aequatio $Z + \int Y \partial y = \text{Const.}$ quam esse integrale quaesitum vel ex ipsa differentiatione patebit. Cum enim sit

$$\partial Z = P \partial x + \partial y \left(\frac{\partial Z}{\partial y}\right) = P \partial x + \partial y \int \partial x \left(\frac{\partial P}{\partial y}\right),$$

erit aequationis inuentae differentiale

$$P \partial x + \partial y \int \partial x \left(\frac{\partial P}{\partial y}\right) + Y \partial y = 0,$$

sed $Y = Q - \int \partial x \left(\frac{\partial P}{\partial y}\right)$, vnde prodit $P \partial x + Q \partial y = 0$ quae est ipsa aequatio differentialis proposita. Quod autem sit $Q - \int \partial x \left(\frac{\partial P}{\partial y}\right)$ functio ipsius y tantum, inde sequitur, quoniam aequatio differentialis per se est integrabilis.

Theorema.

459. Pro omni aequatione, quae per se non est integrabilis semper datur quantitas, per quam ea multiplicata redditur integrabilis.

Demonstratio.

Sit $P \partial x + Q \partial y = 0$ aequatio differentialis, et concipiamus eius integrale completum, quod erit aequatio quaedam

dam inter x et y, in quam conſtans quantitas arbitraria ingrediatur. Ex hac aequatione eruatur haec ipſa conſtans arbitraria, vt prodeat huiusmodi aequatio Conſt. $=$ *functioni cuidam ipſarum* x *et* y, quae differentiata praebeat $0 = M\,\partial x + N\,\partial y$, quae aequatio iam a conſtante illa arbitraria per integrationem ingreſſa eſt libera, ideoque neceſſe eſt vt haec aequatio differentialis conueniat cum propoſita, alioquin integrale ſuppoſitum non eſſet verum. Oportet ergo, vt relatio inter ∂x et ∂y vtrinque prodeat eadem, vnde erit $\frac{P}{Q} = \frac{M}{N}$, ideoque $M = LP$ et $N = LQ$. Sed quia $M\,\partial x + N\,\partial y$ eſt verum differentiale ex differentiatione cuiuspiam functionis ipſarum x et y ortum, eſt $\left(\frac{\partial M}{\partial y}\right) = \left(\frac{\partial N}{\partial x}\right)$. Quare pro aequatione $P\,\partial x + Q\,\partial y = 0$ dabitur certo quidam multiplicator L, vt ſit $\left(\frac{\partial . LP}{\partial y}\right) = \left(\frac{\partial . LQ}{\partial x}\right)$, ſeu vt aequatio per L multiplicata fiat per ſe integrabilis.

Corollarium 1.

460. Pro omni ergo aequatione $P\,\partial x + Q\,\partial y = 0$ datur eiusmodi functio L, vt ſit $\left(\frac{\partial . LP}{\partial y}\right) = \left(\frac{\partial . LQ}{\partial x}\right)$, ideoque euoluendo:

$$L\left(\frac{\partial P}{\partial y}\right) + P\left(\frac{\partial L}{\partial y}\right) = L\left(\frac{\partial Q}{\partial x}\right) + Q\left(\frac{\partial L}{\partial x}\right) \text{ ſeu}$$

$$L\left[\left(\frac{\partial P}{\partial y}\right) - \left(\frac{\partial Q}{\partial x}\right)\right] = Q\left(\frac{\partial L}{\partial x}\right) - P\left(\frac{\partial L}{\partial y}\right)$$

quae functio L ſi fuerit inuenta, aequatio differentialis $LP\,\partial x + LQ\,\partial y = 0$ per ſe erit integrabilis.

Corollarium 2.

461. In aequatione propoſita loco Q tuto vnitatem ſcribere licet, quia omnis aequatio hac forma $P\,\partial x + \partial y = 0$ repraeſentari poteſt. Hinc inuentio multiplicatoris L, qui eam reddat per ſe integrabilem, pendet a reſolutione huius aequationis:

L

$$L\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial L}{\partial x}\right) - P\left(\frac{\partial L}{\partial y}\right),$$

vbi notandum eſt eſſe

$$\partial L = \partial x\left(\frac{\partial L}{\partial x}\right) + \partial y\left(\frac{\partial L}{\partial y}\right).$$

Scholion.

462. Quoniam hic quaeritur functio binarum variabilium x et y, quarum relatio mutua minime ſpectatur, quam inuoluit aequatio $P\partial x + Q\partial y = 0$, haec inueſtigatio in noſtrum librum ſecundum incurrit, vbi huiusmodi functio ex data quadam differentialium relatione indagare debet. In hac enim inueſtigatione non attendimus ad aequationem propoſitam, qua formula $P\partial x + Q\partial y$ nihilo aequalis reddi debet, ſed abſolute quaeritur multiplicator L, per quam formula $P\partial x + Q\partial y$ multiplicata abeat in verum differentiale cuiuspiam functionis finitae, quae ſit Z, ita vt habetur $\partial Z = LP\partial x + LQ\partial y$. Quo multiplicatore L inuento tum demum aequalitas $P\partial x + Q\partial y = 0$ ſpectatur, indeque concluditur functionem Z quantitati conſtanti aequari oportere. Cum igitur minime expectari queat, vt methodum tradamus huiusmodi multiplicatores pro quauis aequatione differentiali propoſita inueniendi, eos caſus percurramus, quibus talis multiplicator conſtat, vndecunque ſit repertus. Interim tamen ad pleniorem vſum huius methodi notaſſe iuuabit, ſtatim atque vnum multiplicatorem pro quapiam aequatione differentiali cognouerimus, ex eo facile innumerabiles alios deduci poſſe, qui pariter aequationem propoſitam per ſe integrabilem reddant.

Problema 60.

463. Dato vno multiplicatore L qui aequationem $P\partial x + Q\partial y = 0$ per ſe integrabilem reddat, inuenire innumerabiles alios multiplicatores, qui idem officium praeſtent.

Solu-

Solutio.

Cum ergo $L(P\,\partial x + Q\,\partial y)$ ſit differentiale verum cuiuspiam functionis Z, quaeratur per ſuperiora praecepta haec functio Z, ita vt ſit $L(P\,\partial x + Q\,\partial y) = \partial Z$: et nunc manifeſtum eſt, hanc formulam ∂Z integrationem etiam eſſe admisſuram, ſi per functionem quamcunque ipſius Z quam ita $\Phi : Z$ indicemus, multiplicetur. Cum igitur etiam integrabilis ſit haec formula $(P\,\partial x + Q\,\partial y)\,L\,\Phi : Z$, erit quoque $L\,\Phi : Z$ multiplicator aequationis propoſitae $P\,\partial x + Q\,\partial y = 0$, qui eam reddat integrabilem. Quare inuento vno multiplicatore L, quaeratur per integrationem $Z = \int L(P\,\partial x + Q\,\partial y)$, ac tum expreſſio $L\,\Phi : Z$, vbi pro $\Phi : Z$ functio quaecunque ipſius Z asſumi poteſt, dabit infinitos alios multiplicatores idem officium praeſtantes.

Scholion.

464. Tametſi ſufficiat pro quauis aequatione differentiali vnicum multiplicatorem cognouiſſe, tamen occurrunt caſus, quibus perquam vtile eſt, plures imo infinitos multiplicatores in promtu habere. Veluti ſi aequatio propoſita in duas partes commode diſcerpatur, huiusmodi $(P\,\partial x + Q\,\partial y) + (R\,\partial x + S\,\partial y) = 0$ atque omnes multiplicatores conſtent, quibus vtraque pars ſeorſim $P\,\partial x + Q\,\partial y$ et $R\,\partial x + S\,\partial y$ reddatur integrabilis, inde interdum communis multiplicator vtramque integrabilem reddens concludi poteſt. Sit enim $L\,\Phi : Z$ expreſſio generalis pro omnibus multiplicatoribus formulae $P\,\partial x + Q\,\partial y$ et $M\,\Phi : V$ expreſſio generalis pro omnibus multiplicatoribus formulae $R\,\partial x + S\,\partial y$, et quoniam $\Phi : Z$ et $\Phi : V$ functiones quascunque quantitatum Z et V denotant, ſi eas ita capere liceat, vt fiat $L\,\Phi : Z = M\,\Phi : V$ habebitur multiplicator idoneus pro aequatione $P\,\partial x + Q\,\partial y + R\,\partial x + S\,\partial y = 0$. Intelligitur autem hoc iis tantum caſibus praeſtari poſſe, quibus multipli-

cator pro tota aequatione, etiam singulas eius partes seorsim sumtas integrabiles reddat. Quare cauendum est, ne huic methodo nimium tribuatur, et quando ea non succedit, aequatio pro irresolubili habeatur, euenire enim vtique potest, vt tota aequatio habeat multiplicatorem, qui singulis eius partibus non conueniat. Ita proposita aequatione $P\,\partial x + Q\,\partial y = 0$, multiplicator partem $P\,\partial x$ seorsim integrabilem reddens manifesto est $\frac{X}{P}$, denotante X functionem quamcunque ipsius x, et multiplicator partem alteram $Q\,\partial y$ integrabilem reddens est $\frac{Y}{Q}$: etiamsi autem neutiquam fieri possit, vt sit $\frac{X}{P} = \frac{Y}{Q}$ seu $\frac{P}{Q} = \frac{X}{Y}$, nisi casibus per se obuiis, tamen tota formula $P\,\partial x + Q\,\partial y$ certo semper habet multiplicatorem, quo ea integrabilis reddatur.

Exemplum 1.

465. *Inuenire omnes multiplicatores, quibus formula $\alpha y\,\partial x + \beta x\,\partial y$ integrabilis redditur.*

Primus multiplicator sponte se offert $\frac{1}{xy}$, qui praebet $\frac{\alpha\,\partial x}{x} + \frac{\beta\,\partial y}{y}$, cuius integrale est $\alpha\, l\, x + \beta\, l\, y = l\, x^\alpha y^\beta$. Huius ergo functio quaecunque $\Phi : x^\alpha y^\beta$ in $\frac{1}{xy}$ ducta, dabit multiplicatorem idoneum, cuius itaque forma generalis est $\frac{1}{xy}\Phi : x^\alpha y^\beta$. Functio enim quantitatis $x^\alpha y^\beta$ etiam est functio logarithmi eiusdem quantitatis. Nam si P fuerit functio ipsius p, et Π functio ipsius P, etiam Π est functio ipsius p et vicissim.

Corollarium.

466. Si pro functione sumatur potestas quaecunque $x^{n\alpha} y^{n\beta}$, formula $\alpha y\,\partial x + \beta x\,\partial y$ integrabilis redditur, si multiplicetur per $x^{n\alpha-1} y^{n\beta-1}$, quo quidem casu integrale sponte patet, est enim $x^{n\alpha} y^{n\beta}$.

Exem-

Exemplum 2.

467. *Inuenire omnes multiplicatores, qui hanc formulam integrabilem* $Xy\,\partial x + \partial y$ *reddant.*

Primus multiplicator $\frac{1}{y}$ sponte se offert, vnde cum sit $\int(X\,\partial x + \frac{\partial y}{y}) = \int X\,\partial x + ly$ seu $l\,e^{\int P\,\partial x}y$, omnes functiones huius quantitatis, seu huius $e^{\int P\,\partial x}y$ per y diuisae, dabunt multiplicatores idoneos. Vnde expressio generalis pro omnibus multiplicatoribus erit $= \frac{1}{y}\Phi : \int e^{\int X\,\partial x}y$.

Corollarium.

468. Pro formula $Xy\,\partial x + \partial y$ multiplicator quoque est $e^{\int X\,\partial x}$, qui est functio ipsius x tantum; quo ergo cum etiam formula $\mathfrak{X}\,\partial x$, denotante $\mathfrak{X}$ functionem quamcunque ipsius x, integrabilis reddatur, ille multiplicator etiam huic formulae $\partial y + Xy\,\partial x + \mathfrak{X}\,\partial x$ conueniet.

Problema 61.

469. Proposita aequatione $\partial y + Xy\,\partial x = \mathfrak{X}\,\partial x$, in qua X et $\mathfrak{X}$ sint functiones quaecunque ipsius x, inuenire multiplicatorem idoneum, eamque integrare.

Solutio.

Cum alterum membrum $\mathfrak{X}\,\partial x$ per functionem quamcunque ipsius x multiplicatum fiat integrabile, dispiciatur num etiam prius membrum $\partial y + Xy\,\partial x$ per huiusmodi multiplicatorem integrabile reddi possit. Quod cum praestet multiplicator $e^{\int X\,\partial x}$, hoc adhibito habebitur aequatio integralis quaesita

$$e^{\int X\,\partial x}y = \int e^{\int X\,\partial x}\mathfrak{X}\,\partial x, \text{ siue}$$

$$y = e^{-\int X\,\partial x}\int e^{\int X\,\partial x}\mathfrak{X}\,\partial x,$$

vti iam supra inuenimus.

Corollarium 1.

470. Patet etiam si loco y adsit functio quaecunque ipsius y, vt habeatur haec aequatio $\partial Y + YX\partial x = \mathfrak{X}\partial x$, eam per multiplicatorem $e^{\int X\partial x}$ reddi integrabilem, et integrale fore:

$$e^{\int X\partial x} Y = \int e^{\int X\partial x} \mathfrak{X}\partial x.$$

Corollarium 2.

471. Quare etiam haec aequatio $\partial y + yX\partial x = y^n\mathfrak{X}\partial x$, quia per y^n diuisa abit in $\frac{\partial y}{y^n} + \frac{X\partial x}{y^{n-1}} = \mathfrak{X}\partial x$, vbi posito $\frac{1}{y^{n-1}} = Y$, ob $-\frac{(n-1)\partial y}{y^n} = \partial Y$, seu $\frac{\partial y}{y^n} = -\frac{\partial Y}{n-1}$, prodit $-\frac{\partial Y}{n-1} + YX\partial x = \mathfrak{X}\partial x$, seu

$$\partial Y - (n-1)YX\partial x = -(n-1)\mathfrak{X}\partial x,$$

qui per multiplicatorem $e^{-(n-1)\int X\partial x}$ fit integrabilis: eiusque integrale erit

$$e^{-(n-1)\int X\partial x} Y = -(n-1)\int e^{-(n-1)\int X\partial x}\mathfrak{X}\partial x, \text{ siue}$$

$$\frac{1}{y^{n-1}} = -(n-1)\, e^{(n-1)\int X\partial x}\int e^{-(n-1)\int X\partial x}\mathfrak{X}\partial x.$$

Scholion.

472. Cum pro membro $\partial y + yX\partial x$ multiplicator generalis sit $\frac{1}{y}\Phi : e^{\int X\partial x} y$, sumta loco functionis potestate, multiplicator idoneus erit $e^{m\int X\partial x} y^{m-1}$, integrale praebens $\frac{1}{m} e^{m\int X\partial x} y^{m-1}$. Efficiendum ergo est, vt etiam idem multiplicator alterum membrum $y^n\mathfrak{X}\partial x$ reddat integrabile; quod euenit sumendo $m - 1 = -n$, seu $m = 1 - n$, ex quo huius membri integrale fit $\int e^{m\int X\partial x}\mathfrak{X}\partial x$, ita vt aequatio integralis quaesita obtineatur:

$$\frac{1}{1-n} e^{(1-n)\int X \partial x} y^{1-n} = \int e^{(1-n)\int X \partial x} \mathfrak{X} \partial x,$$

quae cum modo inuenta prorſus congruit.

Problema 62.

473. Propoſita aequatione differentiali

$$\alpha y \partial x + \beta x \partial y = x^m y^n (\gamma y \partial x + \delta x \partial y),$$

inuenire multiplicatorem idoneum, qui eam integrabilem reddat, ipſumque integrale aſſignare.

Solutio.

Conſideretur vtrumque membrum ſeorſim; ac pro priori vidimus $\alpha y \partial x + \beta x \partial y$ omnes multiplicatores idoneos contineri in hac forma $\frac{1}{xy} \Phi : x^\alpha y^\beta$. Pro altera parte

$$x^m y^n (\gamma y \partial x + \delta x \partial y),$$

primus multiplicator eſt $\frac{1}{x^{m+1} y^{n+1}}$, quo prodit $\frac{\gamma \partial x}{x} + \frac{\delta \partial y}{y}$, cuius integrale eſt $l\, x^\gamma y^\delta$: ergo forma generalis pro eius multiplicatoribus eſt $\frac{1}{x^{m+1} y^{n+1}} \Phi : x^\gamma y^\delta$. Quo nunc hi duo multiplicatores pares reddantur, loco functionum ſumantur poteſtates, fiatque

$$x^{\mu\alpha - 1} y^{\mu\beta - 1} = x^{\nu\gamma - m - 1} y^{\nu\delta - n - 1},$$

vnde ſtatui oportet $\mu\alpha = \nu\gamma - m$ et $\mu\beta = \nu\delta - n$; hincque colligitur:

$$\mu = \frac{\gamma n - \delta m}{\alpha\delta - \beta\gamma} \text{ et } \nu = \frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma}.$$

Quocirca multiplicator erit

$$x^{\mu\alpha - 1} y^{\mu\beta - 1} = x^{\nu\gamma - m - 1} y^{\nu\delta - n - 1},$$

vnde aequatio noſtra induit hanc formam

$$x^{\mu\alpha - 1} y^{\mu\beta - 1} (\alpha y \partial x + \beta x \partial y) = x^{\nu\gamma - 1} y^{\nu\delta - 1} (\gamma y \partial x + \delta x \partial y);$$

 vbi

vbi vtrumque membrum per se est integrabile, ideoque integrale quaesitum

$$\frac{1}{\mu} x^{\mu\alpha} y^{\mu\beta} = \frac{1}{\nu} x^{\nu\gamma} y^{\nu\delta} + \text{Const.}$$

quod conuenit cum eo, quod capite praecedente est inuentum.

Corollarium 1.

474. Posito ergo breuitatis gratia

$$\mu = \frac{\gamma n - \delta m}{\alpha\delta - \beta\gamma} \text{ et } \nu = \frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma},$$

aequationis differentialis

$$\alpha y \partial x + \beta x \partial y = x^m y^n (\gamma y \partial x + \delta x \partial y)$$

integrale completum est

$$\frac{1}{\mu} x^{\mu\alpha} y^{\mu\beta} = \frac{1}{\nu} x^{\nu\gamma} y^{\nu\delta} + \text{Const.}$$

Corollarium 2.

475. Si eueniat, vt sit $\mu = 0$, seu $\gamma n = \delta m$, integrale ad logarithmos reducetur, eritque

$$l\, x^\alpha y^\beta = \frac{1}{\nu} x^{\nu\gamma} y^{\nu\delta} + \text{Const.}$$

Sin autem sit $\nu = 0$, seu $\alpha n = \beta m$, erit integrale

$$\frac{1}{\mu} x^{\mu\alpha} y^{\mu\beta} = l\, x^\gamma y^\delta + \text{Const.}$$

Scholion.

476. Hinc autem casus excipi videntur, quo $\alpha\delta = \beta\gamma$, quia tum ambo numeri μ et ν fiunt infiniti. Verum si $\delta = \frac{\beta\gamma}{\alpha}$ aequatio nostra hanc induit formam

$$\alpha y \partial x + \beta x \partial y = \frac{\gamma}{\alpha} x^m y^n (\alpha y \partial x + \beta x \partial y), \text{ seu}$$

$$(\alpha y \partial x + \beta x \partial y)(1 - \frac{\gamma}{\alpha} x^m y^n) = 0,$$

quae cum habeat duos factores, duplex solutio ex vtroque seorsim ad nihilum reducto, deriuatur. Prior scilicet nascitur ex $\alpha y \partial x + \beta x \partial y = 0$, cuius integrale est $x^\alpha y^\beta = \text{Const.}$ alter

vero

vero factor per se dat aequationem finitam $1 - \frac{\gamma}{\alpha} x^m y^n = 0$, quarum solutionum vtraque satisfacit. Atque hoc in genere tenendum est de omnibus aequationibus differentialibus, quas in factores resoluere licet, vbi perinde atque in aequationibus finitis singuli factores praebent solutiones. Plerumque autem factores finiti statim, antequam integratio suscipitur, per divisionem tolli solent, quandoquidem non ex natura rei, sed per operationes institutas demum accessisse censentur, ita vt perinde ac in Algebra saepe fieri solet, ad solutiones inutiles essent perducturi.

Problema 63.

477. Proposita aequatione differentiali homogenea, multiplicatorem idoneum inuenire, qui eam integrabilem reddat, indeque eius integrale eruere.

Solutio.

Sit $P\partial x + Q\partial y = 0$ aequatio proposita, in qua P et Q sint functiones homogeneae n dimensionum ipsarum x et y, ac quaeramus multiplicatorem L, qui sit etiam functio homogenea, cuius dimensionum numerus sit λ. Cum iam formula $L(P\partial x + Q\partial y)$ sit integrabilis, erit integrale functio $\lambda + n + 1$ dimensionum ipsarum x et y, quae functio si ponatur Z, erit ex natura functionum homogenearum

$$LPx + LQy = (\lambda + n + 1)Z.$$

Quare si λ sumatur $= -n - 1$, quantitas $LPx + LQy$ erit vel $= 0$, vel constans, vnde obtinemus $L = \frac{1}{Px + Qy}$, qui ergo est multiplicator idoneus pro nostra aequatione. Idem quoque ex separatione variabilium colligitur: posito enim $y = ux$, fiet $P = x^n U$ et $Q = x^n V$, existentibus U et V functionibus u ipsius tantum, et ob $\partial y = u\partial x + x\partial u$

erit

erit $P\,\partial x + Q\,\partial y = x^n U\,\partial x + x^n V u\,\partial x + x^n V x\,\partial u$;
seu $P\,\partial x + Q\,\partial y = x^n (U + V u)\,\partial x + x^{n+1} V\,\partial u$.

At haec formula per $x^{n+1}(U + Vu)$ diuisa fit integrabilis, ideoque et formula nostra $P\,\partial x + Q\,\partial y$ diuisa per

$$x^{n+1}(U + V u) = P x + Q y,$$

restitutis valoribus $U = \frac{P}{x^n}$, $V = \frac{Q}{x^n}$, et $u = \frac{y}{x}$, fiet integrabilis; seu multiplicator idoneus est $\frac{1}{Px + Qy}$, vnde haec aequatio $\frac{P\,\partial x + Q\,\partial y}{Px + Qy} = 0$, semper per se est integrabilis.

Iam ad integrale ipsius inueniendum, integretur formula $\int \frac{P\,\partial x}{Px + Qy}$ spectando y vt constantem, ac determinetur certa ratione vt euanescat posito $x = f$. Tum posito breuitatis caussa $\frac{P}{Px + Qy} = R$, sumatur valor $(\frac{\partial R}{\partial y})$, et eadem lege quaeratur integrale $\int \partial x (\frac{\partial R}{\partial y})$, spectando iterum y vt constantem. Tum erit $\frac{Q}{Px + Qy} - \int \partial x (\frac{\partial R}{\partial y})$ functio ipsius y tantum seu

$$\frac{Q}{Px + Qy} - \int \partial x \left(\frac{\partial R}{\partial y}\right) = Y:$$

atque hinc erit integrale quaesitum

$$\int \frac{P\,\partial x}{Px + Qy} + \int Y\,\partial y = \text{Const.}$$

Corollarium 1.

478. Cum ergo formula $\frac{P\,\partial x + Q\,\partial y}{Px + Qy}$ sit per se integrabilis, si breuitatis gratia ponamus

$$\frac{P}{Px + Qy} = R \text{ et } \frac{Q}{Px + Qy} = S,$$

necesse est sit $(\frac{\partial R}{\partial y}) = (\frac{\partial S}{\partial x})$. At est

$$\left(\frac{\partial R}{\partial y}\right) = \left[Qy\left(\frac{\partial P}{\partial y}\right) - Py\left(\frac{\partial Q}{\partial y}\right) - PQ\right] : (Px + Qy)^2 \text{ et}$$
$$\left(\frac{\partial S}{\partial x}\right) = \left[Px\left(\frac{\partial Q}{\partial x}\right) - Qx\left(\frac{\partial P}{\partial x}\right) - PQ\right] : (Px + Qy)^2.$$

Quam-

Quamobrem habebitur

$$Q y \left(\frac{\partial P}{\partial y}\right) - P y \left(\frac{\partial Q}{\partial y}\right) = P x \left(\frac{\partial Q}{\partial x}\right) - Q x \left(\frac{\partial P}{\partial x}\right).$$

Corollarium 2.

479. Haec aequalitas etiam ex natura functionum homogenearum concluditur. Cum enim P et Q sint functiones n dimensionum ipsarum x et y, ob

$$\partial P = \partial x \left(\frac{\partial P}{\partial x}\right) + \partial y \left(\frac{\partial P}{\partial y}\right) \text{ et } \partial Q = \partial x \left(\frac{\partial Q}{\partial x}\right) + \partial y \left(\frac{\partial Q}{\partial y}\right) \text{ erit}$$

$$n P = x \left(\frac{\partial P}{\partial x}\right) + y \left(\frac{\partial P}{\partial y}\right) \text{ et } n Q = x \left(\frac{\partial Q}{\partial x}\right) + y \left(\frac{\partial Q}{\partial y}\right).$$

Aequalitas autem inuenta est

$$Q \left[x \left(\frac{\partial P}{\partial x}\right) + y \left(\frac{\partial P}{\partial y}\right)\right] = P \left[x \left(\frac{\partial Q}{\partial x}\right) + y \left(\frac{\partial Q}{\partial y}\right)\right],$$

quae hinc abit in identicam $n P Q = n P Q$.

Corollarium 3.

480. Si aequatio homogenea $P \partial x + Q \partial y = 0$ fuerit per se integrabilis, et P et Q sint functiones -1 dimensionis, erit $P x + Q y$ numerus constans. Veluti cum

$$\frac{x \partial x + y \partial y}{x x + y y} = 0,$$

huiusmodi sit aequatio, si loco ∂x et ∂y scribantur x et y, prodit $\frac{x x + y y}{x x + y y} = 1$.

Scholion.

481. In calculo differentiali ostendimus, si V fuerit functio homogenea n dimensionum ipsarum x et y, ponaturque $\partial V = P \partial x + Q \partial y$, fore $P x + Q y = n V$. Quare si $P \partial x + Q \partial y$ fuerit formula integrabilis, et P et Q functiones homogeneae $n - 1$ dimensionum, integrale statim habetur, erit enim $V = \frac{1}{n}(P x + Q y)$, neque ad hoc vlla integratione est opus. Interim tamen videmus hinc excipi oportere casum quo $n = 0$, vti fit in nostra aequatione per mul-

 tiplica-

tiplicatorem integrabili reddita $\frac{P\partial x + Q\partial y}{Px + Qy} = 0$, vbi ∂x et ∂y multiplicantur per functiones -1 dimensionis, neque enim hic integrale fine integratione obtineri poteſt. Ratio autem huius exceptionis in hoc eſt ſita, quod formulae integrabilis $P\partial x + Q\partial y$, in qua P et Q ſunt functiones homogeneae $n-1$ dimenſionum, integrale tum tantum ſit functio homogenea n dimenſionum, quando n non eſt $=0$, hoc enim ſolo caſu fieri poteſt, vt integrale non ſit functio nullius dimenſionis, quemadmodum fit in hac formula differentiali $\frac{x\partial x + y\partial y}{xx + yy}$, quippe cuius integrale eſt $\frac{1}{2}l(xx+yy)$. Quocirca, quod formula $\frac{P\partial x + Q\partial y}{Px + Qy}$ ſit integrabilis, hoc peculiari modo demonſtrauimus, ex ratione ſeparabilitatis deducto. Interim tamen ſine vllo reſpectu, vnde hoc cognouerimus, id in praeſenti negotio maxime eſt notatu dignum, omnes aequationes homogeneas $P\partial x + Q\partial y = 0$, per multiplicatorem $\frac{1}{Px + Qy}$ per ſe reddi integrabiles. Methodus igitur deſideratur, cuius beneficio hunc multiplicatorem a priori inuenire liceret; qua methodo ſane maxima incrementa in Analyſin importarentur. Quamdiu autem eousque pertingere non licet, plurimum intererit huiusmodi multiplicatores pro pluribus caſibus probe notaſſe; quod cum iam in duobus aequationum generibus praeſtiterimus, pro reliquis aequationibus, quas ſupra integrare docuimus, multiplicatores inueſtigemus; ipſa autem reductio ad ſeparationem nobis hos multiplicatores pateſaciet, vti in ſequente problemate docebimus.

Problema 64.

482. Propoſita aequatione differentiali, quam ad ſeparationem variabilium reducere liceat, inuenire multiplicatorem, per quem ea per ſe integrabilis reddatur.

Solutio.

Solutio.

Sit $P\partial x + Q\partial y = 0$, quae certa quadam ſubſtitutione, dum loco x et y aliae binae variabiles t et u introducuntur, ad ſeparationem accommodetur: ponamus ergo facta hac ſubſtitutione fieri $P\partial x + Q\partial y = R\partial t + S\partial u$, nunc autem hanc formulam $R\partial t + S\partial u$ ſi per V diuidatur, ſeparari, ita vt in hac formula $\frac{R\partial t + S\partial u}{V}$ quantitas $\frac{R}{V}$ ſit functio ſolius t, et $\frac{S}{V}$ functio ſolius u. Cum igitur formula $\frac{R\partial t + S\partial u}{V}$ per ſe ſit integrabilis, etiam integrabilis erit haec $\frac{P\partial x + Q\partial y}{V}$ quippe illi aequalis, ſiquidem in V variabiles x et y reſtituantur. Hinc ergo ex reductione ad ſeparabilitatem aequationis $P\partial x + Q\partial y = 0$ diſcimus, multiplicatorem quo ea integrabilis reddatur, eſſe $\frac{1}{V}$, ſicque quas aequationes ad ſeparationem variabilium perducere licet, pro iisdem multiplicatorem, qui illas integrabiles reddat, aſſignare poſſumus.

Corollarium 1.

483. Methodus ergo per multiplicatores integrandi aequationes differentiales aeque late patet ac prior methodus, ope ſeparationis variabilium; propterea quod ipſa ſeparatio pro quauis aequatione, vbi ſuccedit, multiplicatorem ſuppeditat.

Corollarium 2.

484. Contra autem methodus per multiplicatores integrandi latius patet altera, ſi pro eiusmodi aequationibus multiplicatores aſſignare liceat, quas quomodo ad ſeparationem perduci debeant, non conſtet.

Scholion.

485. Etſi autem ex reductione ad ſeparationem idoneum multiplicatorem elicere licet, tamen nondum intelligitur,

 quo-

quomodo cognito multiplicatore, ſeparatio variabilium inſtitui debeat, quare etiam ob hanc rationem methodus per multiplicatores integrandi alteri longe praeferenda videtur. Quamvis enim hactenus ipſa ſeparatio nos ad inuentionem multiplicatorum perduxerit, nullum tamen eſt dubium quin detur via multiplicatores inueniendi, nullo reſpectu ad ſeparationem habito, licet haec via etiamnum nobis ſit incognita. Ea autem paullatim planior reddetur, ſi pro quamplurimis aequationibus multiplicatores idoneos cognouerimus, ex quo quos adhuc ex ſeparatione eruere licet, indagemus in ſubiunctis exemplis.

Exemplum 1.

486. *Propoſita aequatione differentiali primi ordinis*

$$\partial x(\alpha x+\beta y+\gamma)+\partial y(\delta x+\varepsilon y+\zeta)=0,$$

pro ea multiplicatorem idoneum aſſignare.

Haec aequatio ad ſeparationem praeparatur ponendo primo

$$\alpha x+\beta y+\gamma=r \text{ et } \delta x+\varepsilon y+\zeta=s,$$

ideoque

$$\alpha\partial x+\beta\partial y=\partial r \text{ et } \delta\partial x+\varepsilon\partial y=\partial s,$$

vnde oritur

$$\partial x=\frac{\varepsilon\partial r-\beta\partial s}{\alpha\varepsilon-\beta\delta} \text{ et } \partial y=\frac{\alpha\partial s-\delta\partial r}{\alpha\varepsilon-\beta\delta},$$

hincque aequatio noſtra omiſſo denominatore vtpote conſtante, erit

$$\varepsilon r\partial r-\beta r\partial s+\alpha s\partial s-\delta s\partial r=0,$$

quae cum ſit homogenea, per $\varepsilon rr-(\beta+\delta)sr+\alpha ss$ diuiſa, fit integrabilis. Quod idem ex ſeparatione colligitur, poſito enim $r=su$, prodit

$$\varepsilon ssu\partial u+\varepsilon suu\partial s-\beta su\partial s+\alpha s\partial s-\delta ss\partial u-\delta su\partial s=0 \text{ ſeu}$$

$ss\partial u$

$$ss\,\partial u(\varepsilon u - \delta) + s\,\partial s(\varepsilon uu - \beta u - \delta u + \alpha) = 0,$$

quae diuiſa per $ss(\varepsilon uu - \beta u - \delta u + \alpha)$ ſeparatur. Quare multiplicator noſtrae aequationis propoſitae eſt

$$\frac{1}{ss(\varepsilon uu - \beta u - \delta u + \alpha} = \frac{1}{\varepsilon rr - \beta rs - \delta rs + \alpha ss} = \frac{1}{r(\varepsilon r - \beta s) + s(\alpha s - \delta r)},$$

qui reſtitutis valoribus fit

$$\frac{1}{(\alpha x + \beta y + \gamma)[(\alpha\varepsilon - \beta\delta)x + \gamma\varepsilon - \beta\zeta] + (\delta x + \varepsilon y + \zeta)[(\alpha\varepsilon - \beta\delta)y + \alpha\zeta - \gamma\delta]}:$$

atque euolutione facta

$$\frac{1}{\begin{array}{r}(\alpha\varepsilon - \beta\delta)[\alpha xx + (\beta + \delta)xy + \varepsilon yy + \gamma x + \zeta y] + \alpha\zeta\zeta - (\beta - \delta)\gamma\zeta + \gamma\gamma\varepsilon \\ + [\alpha\gamma\varepsilon - (\beta - \delta)\alpha\zeta - \gamma\delta\delta]\,x + [\alpha\varepsilon\zeta + (\beta - \delta)\gamma\varepsilon - \beta\beta\zeta]\,y.\end{array}}$$

Quare per ſe integrabilis erit haec aequatio

$$\frac{\partial x(\alpha x + \beta y + \gamma) + \partial y(\delta x + \varepsilon y + \zeta)}{(\alpha\varepsilon - \beta\delta)[\alpha xx + (\beta + \delta)xy + \varepsilon yy + \gamma x + \zeta y] + Ax + By + C}:$$

exiſtente

$$A = \alpha\gamma\varepsilon - (\beta - \delta)\alpha\zeta - \gamma\delta\delta$$
$$B = \alpha\varepsilon\zeta + (\beta - \delta)\gamma\varepsilon - \beta\beta\zeta$$
$$C = \alpha\zeta\zeta - (\beta - \delta)\gamma\zeta + \gamma\gamma\varepsilon.$$

Corollarium.

487. Etiamſi forte fiat $\alpha\varepsilon - \beta\delta = 0$, hic multiplicator non turbatur, cum tamen ſeparatio non ſuccedat hac quidem operatione. Sit enim $\alpha = ma$, $\beta = mb$, $\delta = na$, $\varepsilon = nb$, vt habeatur haec aequatio

$$\partial x\,[m(ax + by) + \gamma] + \partial y\,[n(ax + by) + \zeta] = 0$$

ob $A = a(na - mb)(m\zeta - n\gamma)$
$B = b(na - mb)(m\zeta - n\gamma)$ et
$C = (m\zeta - n\gamma)(a\zeta - b\gamma)$,

omiſſo factore communi, multiplicator eſt

$$\frac{1}{(na - mb)(ax + by) + a\zeta - b\gamma},$$

 ita

ita vt haec aequatio per se sit integrabilis

$$\frac{(ax+by)(m\partial x+n\partial y)+\gamma\partial x+\zeta\partial y}{(na-mb)(ax+by)+a\zeta-\beta\gamma}=0.$$

Exemplum 2.

488. *Proposita aequatione differentiali*

$$y\partial x(c+nx)-\partial y(y+a+bx+nxx)=0,$$

multiplicatorem idoneum inuenire.

Fiat substitutio $\frac{y(c+nx)}{y+a+bx+nxx}=u$, seu $y=\frac{u(a+bx+nxx)}{c+nx-u}$, vt contrahatur aequatio nostra in hanc formam

$$y\partial x(c+nx)-\frac{y\partial y(c+nx)}{u}=0,$$

seu $\frac{y(c+nx)}{u}(u\partial x-\partial y)=0$, vel $\frac{yy(c+nx)}{u}\left(\frac{\partial y}{y}-\frac{u\partial x}{y}\right)=0$; probe enim cauendum est, ne hic vllus factor omittatur. At facta substitutione reperitur

$$\frac{\partial y}{y}-\frac{u\partial x}{y}=\frac{\partial u}{u}+\frac{\partial x(b+2nx)}{a+bx+nxx}+\frac{\partial u-n\partial x}{c+nx-u}-\frac{\partial x(c+nx-u)}{a+bx+nxx}$$
$$=\frac{\partial u(c+nx)}{u(c+nx-u)}-\frac{\partial x(na+cc-bc+(b-2c)u+uu)}{(c+nx-u)(a+bx+nxx)}.$$

Vnde aequatio nostra induet hanc formam

$$\frac{yy(c+nx)^2}{u(c+nx-u)}\left(\frac{\partial u}{u}-\frac{\partial x(na+cc-bc+(b-2c)u+uu)}{(a+bx+nxx)(c+nx)}\right)=0,$$

quae ergo separabitur ducta in hunc multiplicatorem

$$\frac{u(c+nx-u)}{yy(c+nx)^2(na+cc-bc+(b-2c)u+uu)},$$

tum enim prodit

$$\frac{\partial u}{u(na+cc-bc+(b-2c)u+uu)}-\frac{\partial x}{(a+bx+nxx)(c+nx)}=0.$$

Quo igitur multiplicatorem quaesitum consequamur, ibi loco u tantum opus est suum valorem restituere tum autem reperitur multiplicator

$$\frac{a+bx+nxx}{n(a+bx+nxx)y^3+(a+bx+nxx)[2na-bc+n(b-2c)x]yy+(na+cc-bc)(a+bx+nxx)^2y},$$

qui reducitur ad hanc formam

$$\frac{1}{ny^3+(2na-bc)yy+n(b-2c)xyy+(na+cc-bc)(a+bx+nxx)y}.$$

Exem-

Exemplum 3.

489. *Propofita aequatione differentiali*

$$\frac{n\partial x(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}+(x-y)\partial y=0,$$

inuenire multiplicatorem qui eam integrabilem reddat.

Pofuimus fupra (435.) $y=\frac{x-u}{1+xu}$, feu $u=\frac{x-y}{1+xy}$, vnde fit $x-y=\frac{u(1+xx)}{1+xu}$, et $1+yy=\frac{(1+xx)(1+uu)}{(1+xu)^2}$, hincque noftra aequatio hanc induit formam

$$\frac{n\partial x(1+xx)(1+uu)^{\frac{3}{2}}}{(1+xu)^3}+\frac{u\partial x(1+xx)(1+uu)-u\partial u(1+xx)^2}{(1+xu)^3}=0,$$

quae primo multiplicata per $(1+xu)^3$, tum diuifa per

$$(1+xx)^2(1+uu)[u+n\sqrt{(1+uu)}]$$

feparatur. Quare aequationis noftrae multiplicator erit

$$\frac{(1+xu)^3}{(1+xx)^2(1+uu)[u+n\sqrt{(1+uu)}]},$$

qui primo ob $1+uu=\frac{(1+yy)(1+xu)^2}{1+xx}$, abit in

$$\frac{1+xu}{(1+xx)(1+yy)[u+n\sqrt{(1+uu)}]}.$$

Nunc ob $u=\frac{x-y}{1+xy}$, eft $\sqrt{(1+uu)}=\frac{\sqrt{(1+xx)(1+yy)}}{1+xy}$ et $1+xu=\frac{1+xx}{1+xy}$, ideoque nofter multiplicator colligitur:

$$\frac{1}{(1+yy)[x-y+n\sqrt{(1+xx)(1+yy)}]},$$

ita vt per fe fit integrabilis haec aequatio

$$\frac{n\partial x(1+yy)\sqrt{(1+yy)}+(x-y)\partial y\sqrt{(1+xx)}}{(1+yy)[x-y+n\sqrt{(1+xx)(1+yy)}]\sqrt{(1+xx)}}=0,$$

cuius integrationi non immoror, cum iam fupra integrale exhibuerim.

Exemplum 4.

490. *Aliud exemplum memoratu dignum fuppeditat haec aequatio*

$$y\partial x-x\partial y+ax^ny\partial y(x^n+b)^{\frac{1}{n}}=0,$$

quae

quae si hac forma repraesentetur

$$x\,\partial y - y\,\partial x + \tfrac{1}{b}x^{n+1}\partial y = \tfrac{1}{b}x^{n+1}\partial y + a x^n y\,\partial y\,(x^n+b)^{\frac{1}{n}}$$

euenit, vt vtrumque integrabile existat, si ducatur in hunc multiplicatorem

$$\frac{y^{n-1}}{x^{n+1} + a\,b\,x^n y\,(x^n+b)^{\frac{1}{n}}}:$$

ad quem inueniendum ex separatione variabilium, adhibeatur haec substitutio non adeo obuia $\frac{x}{(x^n+b)^{\frac{1}{n}}} = v\,y$, vnde fit $x^n = \frac{b\,v^n y^n}{1 - v^n y^n}$, et hinc aequatio

$$\frac{y\,\partial x - x\,\partial y}{(x^n+b)^{\frac{1}{n}}} + a\,x^n y\,\partial y = 0 \text{ abit in hanc}$$

$$\frac{y\,y\,\partial v + v^{n+1}y^{n+1}\partial y + a\,b\,v^n y^{n+1}\partial y}{1 - v^n y^n} = 0,$$

quae multiplicata per $\frac{1 - v^n y^n}{y\,y\,v^n\,(a\,b + v)}$ separatur

$$\frac{\partial v}{v^n\,(a\,b+v)} + y^{n-1}\partial y = 0,$$

vnde idem ille multiplicator colligitur.

Exemplum 5.

491. *Proposita aequatione differentiali*

$$\partial x + y\,y\,\partial x - \tfrac{a\,\partial x}{x^4} = 0$$

inuenire multiplicatorem, quo ea integrabilis reddatur.

Se-

Secundum §. 436. ponatur $x = \frac{1}{t}$ et ob $\partial x = -\frac{\partial t}{tt}$, noſtra formula erit $\partial y - \frac{yy\partial t}{tt} + att\partial t$, in qua porro ſtatuatur $y = t - ttz$, et prodibit $-tt(\partial z + zz\partial t - a\partial t)$, quae per $tt(zz-a)$ diuiſa ſeparatur, ergo et noſtra aequatio diuiſa per $tt(zz-a) = \frac{(1-y)^2 - at^4}{tt} = (1-xy)^2 - \frac{a}{xx}$ fiet integrabilis, ex quo multiplicator erit $= \frac{xx}{xx(1-xy)^2 - a}$, et aequatio per ſe integrabilis $\frac{x^4\partial y + x^4 yy\partial x - a\partial x}{x^4(1-xy)^2 - axx} = 0$. Spectetur iam x vt conſtans, eritque ex ∂y natum integrale

$$\frac{1}{2\sqrt{a}}\, l\, \frac{\sqrt{a} + x(1-xy)}{\sqrt{a} - x(1-xy)} + X,$$

pro quo vt valor ipſius X obtineatur, differentietur denuo, ac prodibit

$$\frac{2xy\partial x - \partial x}{xx(1-xy)^2 - a} + \partial X = \frac{x^4 yy\partial x - a\partial x}{x^4(1-xy)^2 - axx};$$

vnde

$$\partial X = \frac{x^4 yy\partial x - a\partial x - 2x^3 y\partial x + xx\partial x}{x^4(1-xy)^2 - axx} = \frac{\partial x}{xx},$$

et $X = -\frac{1}{x} + C$; quare aequatio integralis completa erit

$$l\, \frac{\sqrt{a} + x(1-xy)}{\sqrt{a} - x(1-xy)} = \frac{2\sqrt{a}}{x} + C.$$

Scholion.

492. En ergo plures caſus aequationum differentialium pro quibus multiplicatores nouimus, ex quorum contemplatione haec inſignis inueſtigatio non parum adiuuari videtur. Quanquam autem adhuc longe abſumus a certa methodo, pro quouis caſu multiplicatores idoneos inueniendi; hinc tamen formas aequationum colligere poterimus, vt per datos multiplicatores integrabiles reddantur; quod negotium cum in hac ardua doctrina maximam vtilitatem allaturum videatur, in ſequente capite aequationes inueſtigabimus, quibus dati multiplicatores conueniant? exempla ſcilicet hic euoluta idoneas multiplicatorum formas nobis ſuppeditant, quibus noſtram inueſtigationem ſuperſtruere licebit.

CAPVT III.

DE INVESTIGATIONE AEQVATIONVM DIFFERENTIALIVM QVAE PER MVLTIPLICATORES DATAE FORMAE INTEGRABILES REDDANTVR.

Problema 65.

493.

Definire functiones P et Q ipsius x, vt aequatio differentialis $Py\partial x + (y+Q)\partial y = 0$, per multiplicatorem $\frac{1}{y^3+Myy+Ny}$, vbi M et N sunt functiones ipsius x, fiat integrabilis.

Solutio.

Necesse igitur est, vt factoris ipsius ∂x, qui est $\frac{Py}{y^3+Myy+Ny}$, differentiale ex variabilitate ipsius y natum, aequale sit differentiali factoris ipsius ∂y, qui est $\frac{y+Q}{y^3+Myy+Ny}$, dum sola x variabilis sumitur. Horum valorum aequalium, neglecto denominatore communi, aequalitas dat

$$-2Py^3 - PMy^2 = (y^3+Myy+Ny)\frac{\partial Q}{\partial x} - (y+Q)\frac{(yy\partial M + y\partial N)}{\partial x},$$

quae secundum potestates ipsius y ordinata praebet

$$\begin{aligned}
0 = {} & 2Py^3\partial x + PMy^2\partial x \\
& + y^3\partial Q \quad + My^2\partial Q + Ny\partial Q \\
& - y^3\partial M \quad - y^2\partial N \\
& \qquad\qquad - Qy^2\partial M - Qy\partial N
\end{aligned}$$

vnde singulis potestatibus seorsim ad nihilum perductis, nanciscimur primo $N\partial Q - Q\partial N = 0$, seu $\frac{\partial N}{N} = \frac{\partial Q}{Q}$, ex cuius inte-

integratione sequitur $N = \alpha Q$. Tum binae reliquae conditiones sunt,

$$\text{I. } 2 P \partial x + \partial Q - \partial M = 0 \text{ et}$$

$$\text{II. } P M \partial x + M \partial Q - \alpha \partial Q - Q \partial M = 0;$$

vnde I. M — II. 2 suppeditat

$$- M \partial Q - M \partial M + 2 \alpha \partial Q + 2 Q \partial M = 0, \text{ seu}$$

$$\partial Q + \frac{2 Q \partial M}{2 \alpha - M} = \frac{M \partial M}{2 \alpha - M},$$

quae per $(2 \alpha - M)^2$ diuisa et integrata dat

$$\frac{Q}{(2 \alpha - M)^2} = \int \frac{M \partial M}{(2 \alpha - M)^3} = - \int \frac{\partial M}{(2 \alpha - M)^2} + 2 \alpha \int \frac{\partial M}{(2 \alpha - M)^3},$$

seu

$$\frac{Q}{(2 \alpha - M)^2} = \frac{-1}{2 \alpha - M} + \frac{\alpha}{(2 \alpha - M)^2} + \beta = \frac{M - \alpha}{(2 \alpha - M)^2} + \beta.$$

Erit ergo

$$Q = M - \alpha + \beta (2 \alpha - M)^2,$$

hincque

$$2 P \partial x = \partial M - \partial Q = + 2 \beta \partial M (\alpha - M):$$

sicque pro M functionem quamcunque ipsius x sumere licet. Capiatur ergo $M = 2 \alpha - X$, erit $P \partial x = - \beta X \partial X$, et $Q = \alpha - X + \beta X X$, atque $N = \alpha \alpha - \alpha X + \alpha \beta X X$. Quocirca pro hac aequatione

$$- \beta y X \partial X + \partial y (\alpha - X + \beta X X + y) = 0$$

habemus hunc multiplicatorem

$$\frac{1}{y^3 + (2 \alpha - X) y y + \alpha (\alpha - X + \beta X X) y},$$

quo ea integrabilis redditur.

Corollarium 1.

494. Tribuatur aequationi haec forma

$$\partial y (y + A + B V + C V V) - C y V \partial V = 0$$

 erit-

eritque $\alpha = A$; $X = -BV$; $\beta XX = \beta BBVV = CVV$: ergo $\beta = \frac{C}{BB}$, vnde multiplicator fiet

$$\frac{1}{y^3 + (2A + BV)yy + A(A + BV + CVV)y}.$$

Corollarium 2.

495. Si hic fumatur $V = a + x$, obtinebitur aequatio fimilis illi, quam fupra §. 488. integrauimus, et multiplicator quoque cum eo, quem ibi dedimus, conuenit. Hic autem multiplicator commodius hac forma exhibetur

$$\frac{1}{y(y+A)^2 + BVy(y+A) + ACVVy}.$$

Corollarium 3.

496. Si ponamus $y + A = z$, noftra aequatio erit

$$\partial z(z + BV + CVV) - C(z - A)V\partial V = 0,$$

cui conuenit multiplicator $\frac{1}{(z-A)(zz+BVz+ACVV)}$; ita vt per fe integrabilis fit haec aequatio

$$\frac{\partial z(z + BV + CVV) - C(z-A)V\partial V}{(z-A)(zz+BVz+ACVV)} = 0.$$

Scholion.

497. Quemadmodum hic aequationis $Py\partial x + (y+Q)\partial y = 0$ multiplicatorem affumfimus $= \frac{y^{-1}}{yy+My+N}$, ita generalius eius loco fumere poterimus $\frac{y^{n-1}}{yy+My+N}$, vt haec aequatio $\frac{Py^n\partial x + (y^n + Qy^{n-1})\partial y}{yy+My+N} = 0$ per fe debeat effe integrabilis, qua comparata cum forma $R\partial x + S\partial y = 0$, vt fit $\left(\frac{\partial R}{\partial y}\right) = \left(\frac{\partial S}{\partial x}\right)$, habebimus

$$(n-2)Py^{n+1} + (n-1)PMy^n + nPNy^{n-1} = (yy+My+N)y^{n-1}\frac{\partial Q}{\partial x}$$
$$-(y^n + Qy^{n-1})\left(\frac{y\partial M}{\partial x} + \frac{\partial N}{\partial x}\right),$$

fiue

ſiue ordinata aequatione

$$\left.\begin{array}{lll} (n-2)Py^{n+1}\partial x & +(n-1)PMy^{n}\partial x & +nPNy^{n-1}\partial x \\ -y^{n+1}\partial Q & -My^{n}\partial Q & -Ny^{n-1}\partial Q \\ +y^{n+1}\partial M & +y^{n}\partial N & +y^{n-1}Q\partial N \\ & +y^{n}Q\partial M & \end{array}\right\} = 0;$$

vnde ſingulis membris ad nihilum reductis, fit

$$\text{I. } (n-2)P\partial x = \partial Q - \partial M$$

$$\text{II. } (n-1)MP\partial x = M\partial Q - Q\partial M - \partial N$$

$$\text{III. } nNP\partial x = N\partial Q - Q\partial N.$$

Sit $P\partial x = \partial V$, eritque ex prima $Q = A + M + (n-2)V$, quo valore in ſecunda ſubſtituto prodit

$$M\partial V + (n-2)V\partial M + A\partial M + \partial N = 0$$

et tertia fit

$$2N\partial V + (n-2)V\partial N + M\partial N - N\partial M + A\partial N = 0:$$

vnde eliminando ∂V reperitur

$$(n-2)V + A = \frac{MM\partial N - MN\partial M - 2N\partial N}{2N\partial M - M\partial N}.$$

Verum ſi hinc vellemus V elidere, in aequationem differentio-differentialem illaberemur. Caſus tamen quo $n = 2$ expediri poteſt.

Exemplum.

498. *Sit in euolutione huius caſus $n = 2$, vt per ſe integrabilis eſſe debeat haec aequatio*

$$\frac{y[Py\partial x + (y+Q)\partial y]}{yy + My + N} = 0.$$

Ac primo eſſe oportet $Q = A + M$, tum vero

$$2AN\partial M - AM\partial N = M(M\partial N - N\partial M) - 2N\partial N,$$

quam ergo aequationem integrare debemus, quae cum in nulla iam tractatarum contineatur, videndum eſt, quomodo tractabilior reddi queat. Ponatur ergo $M = Nu$, vt fiat

 M∂N

$$M \partial N - N \partial M = - N N \partial u, \text{ et}$$
$$2 N \partial M - M \partial N = 2 N N \partial u + N u \partial N, \text{ hinc}$$
$$2 A N N \partial u + A N u \partial N + N^3 u \partial u + N \partial N = 0, \text{ siue}$$
$$\frac{2 \partial N}{N N} + \frac{A u \partial N}{N N} + \frac{2 A \partial u}{N} + u \partial u = 0:$$

statuatur porro $\frac{1}{N} = v$, seu $N = \frac{1}{v}$, habebitur

$$- 2 \partial v - A u \partial v + 2 A v \partial u + u \partial u = 0, \text{ seu}$$
$$\partial v - \frac{2 A v \partial u}{2 + A u} = \frac{u \partial u}{2 + A u}.$$

Vbi variabilis v vnicam habet dimensionem, et hanc ob rem patet, hanc aequationem integrabilem reddi, si diuidatur per $(2 + A u)^2$, prodibitque

$$\frac{v}{(2 + A u)^2} = \int \frac{u \partial u}{(2 + A u)^2} = \frac{C}{A A} - \frac{1 - A u}{A A (2 + A u)^2},$$

ideoque $v = \frac{C (2 + A u)^2 - 1 - A u}{A A}$. Sumto ergo pro u functione quacunque ipsius x, erit

$$N = \frac{A A}{C (2 + A u)^2 - 1 - A u}, \text{ et } M = \frac{A A u}{C (2 + A u)^2 - 1 - A u},$$

atque $Q = \frac{A C (2 + A u)^2 - A}{C (2 + A u)^2 - 1 - A u}$. Iam ex tertia aequatione adipiscimur $2 N P \partial x = N \partial Q - Q \partial N$, seu $2 P \partial x = N \partial . \frac{Q}{N}$, at $\frac{Q}{N} = \frac{C (2 + A u)^2 - 1}{A}$, vnde $\partial . \frac{Q}{N} = 2 C \partial u (2 + A u)$, ideoque

$$P \partial x = \frac{A A C \partial u (2 + A u)}{C (2 + A u)^2 - 1 - A u}.$$

Quocirca aequatio nostra per se integrabilis est

$$\frac{A A C y y \partial u (2 + A u) + y \partial y [C (2 + A u)^2 y - (1 + A u) y + A C (2 + A u)^2 - A]}{C (2 + A u)^2 y y - (1 + A u) y y + A A u y + A A} = 0,$$

quae posito $A u + 2 = t$, induet hanc formam

$$y . \frac{A C y t \partial t + y \partial y (C t t - t + 1) + A \partial y (C t t - 1)}{C t t y y - (t - 1) y y + A (t - 2) y + A A} = 0.$$

Hinc autem posito $A = \alpha$; $C = \frac{\alpha \gamma}{\beta \beta}$ et $t = - \frac{\beta x}{\alpha}$, inuenimus

$$y . \frac{\alpha \gamma x y \partial x + y \partial y (\alpha + \beta x + \gamma x x) - \alpha \partial y (\alpha - \gamma x x)}{(\alpha + \beta x + \gamma x x) y y - \alpha (2 \alpha + \beta x) y + \alpha^2} = 0.$$

Corol-

Corollarium 1.

499. Hoc igitur modo integrari poteſt haec aequatio

$$\alpha\gamma xy\partial x+y\partial y(\alpha+\beta x+\gamma xx)-\alpha\partial y(\alpha-\gamma xx)=0,$$

quae quomodo ad ſeparationem reduci debeat, non ſtatim patet. Eſt autem multiplicator idoneus

$$\frac{y}{(\alpha+\beta x+\gamma xx)yy-\alpha(2\alpha+\beta x)y+\alpha^3}.$$

Corollarium 2.

500. Hic multiplicator etiam hoc modo exprimi poteſt, vt eius denominator in factores reſolvatur

$$\frac{(\alpha+\beta x+\gamma xx)y}{[(\alpha+\beta x+\gamma xx)y-(\alpha+\tfrac{1}{2}\beta x)+\alpha x\surd(\tfrac{1}{4}\beta\beta-\alpha\gamma)][(\alpha+\beta x+\gamma xx)y-\alpha(\alpha+\tfrac{1}{2}\beta x)-\alpha x\surd(\tfrac{1}{4}\beta\beta-\alpha\gamma)]}.$$

Corollarium 3.

501. Si ergo ponamus

$$(\alpha+\beta x+\gamma xx)y-\alpha(\alpha+\tfrac{1}{2}\beta x)=\alpha z,$$

erit multiplicator

$$\frac{\alpha+\tfrac{1}{2}\beta+z}{[z+x\surd(\tfrac{1}{4}\beta\beta-\gamma)]\,[z-x\surd(\tfrac{1}{4}\beta\beta-\alpha\gamma)]}.$$

At ob $y=\dfrac{\alpha\alpha+\tfrac{1}{2}\alpha\beta x+\alpha z}{\alpha+\beta x+\gamma xx}$, aequatio noſtra erit

$$\gamma xy\partial x+\partial y(z+\tfrac{1}{2}\beta x+\gamma xx)=0.$$

At eſt

$$\partial y=\frac{-\tfrac{1}{2}\alpha(\alpha\beta+4\alpha\gamma x+\beta\gamma xx)\partial x-\alpha z\partial x(\beta+2\gamma x)+\alpha\partial z(\alpha+\beta x+\gamma xx)}{(\alpha-\beta x+\gamma xx)^2}$$

hoc autem valore ſubſtituto prodit aequatio nimis complicata.

Pro-

Problema 66.

502. Inuenire aequationem differentialem huius formae

$$y\,P\,\partial x + (Q\,y + R)\,\partial y = 0$$

in qua P, Q et R ſint functiones ipſius x, vt ea integrabilis euadat per hunc multiplicatorem $\frac{y^m}{(1+Sy)^n}$, vbi S eſt etiam functio ipſius x.

Solutio.

Quia ∂x per $\frac{y^{m+1}P}{(1+Sy)^n}$ et ∂y per $\frac{Q\,y^{m+1}+R\,y^m}{(1+Sy)^n}$ multiplicatur, oportet ſit

$$(m+1)\,P\,y^m\,(1+S\,y) - n\,P\,S\,y^{m+1}$$
$$= \frac{(1+S\,y)\,(y^{m+1}\partial Q + y^m\,\partial R) - n\,y\,\partial S\,(Q\,y^{m+1}+R\,y^m)}{\partial x},$$

qua euoluta aequatione erit

$$\left.\begin{matrix} (m+1)\,Py^m\partial x + (m+1-n)\,PSy^{m+1}\partial x - y^{m+2}S\partial Q \\ -y^m\partial R \qquad -y^{m+1}\,\partial Q \qquad\qquad + ny^{m+2}Q\partial S \\ -y^{m+1}\,S\,\partial R \\ +ny^{m+1}\,R\,\partial S \end{matrix}\right\} = 0,$$

hinc fit $P\,\partial x = \frac{\partial R}{m+1}$ et $S\,\partial Q = n\,Q\,\partial S$, ideoque $Q = AS^n$ et $\partial Q = nA\,S^{n-1}\,\partial S$, quibus in membro medio ſubſtitutis fit

$$\tfrac{m+1-n}{m+1}\,S\,\partial R - n\,A\,S^{n-1}\partial S - S\,\partial R + nR\partial S = 0, \text{ ſeu}$$

$$-\tfrac{S\,\partial R}{m+1} - A\,S^{n-1}\,\partial S + R\,\partial S = 0, \text{ ideoque}$$

$$\partial R - \tfrac{(m+1)\,R\,\partial S}{S} = -(m+1)\,A\,S^{n-2}\,\partial S,$$

quae per S^{m+1} diuiſa et integrata praebet

R

$$\frac{R}{S^{m+1}} = B - \frac{(m+1)\,A\,S^{n-m-2}}{n-m-2}.$$

Ponamus $A = (m+2-n)\,C$, vt fit $Q = (m+2-n)\,C\,S^n$, et $R = B\,S^{m+1} + (m+1)\,C\,S^{n-1}$, ideoque

$$P\,\partial x = B\,S^m\,\partial S + (n-1)\,C\,S^{n-2}\,\partial S.$$

Quocirca habebimus hanc aequationem

$$y\,\partial S\,[B\,S^m + (n-1)\,C\,S^{n-2}]$$
$$+\,\partial y\,[(m+2-n)\,C\,S^n y + B\,S^{m+1} + (m+1)\,C\,S^{n-1}] = 0,$$

quae multiplicata per $\frac{y^m}{(1+Sy)^n}$ fit integrabilis, vbi pro S functionem quamcunque ipsius x capere licet.

Corollarium 1.

503. Integrari ergo poterit haec aequatio

$$By\,S^m\partial S + B\,S^{m+1}\partial y + (n-1)\,Cy\,S^{n-2}\partial S + (m+1)\,C\,S^{n-1}\partial y$$
$$+ (m+2-n)\,C\,S^n y\,\partial y = 0,$$

quae sponte resoluitur in has duas partes

$$B\,S^m\,(y\,\partial S + S\,\partial y) +$$
$$C\,S^{n-2}[(n-1)y\,\partial S + (m+1)\,S\,\partial y + (m+2-n)\,S^2 y\,\partial y] = 0,$$

quarum vtraque seorsim per $\frac{y^m}{(1+Sy)^n}$ multiplicata fit integrabilis.

Corollarium 2.

504. Prior pars $B\,S^m\,(y\,\partial S + S\,\partial y)$ integrabilis redditur per hunc multiplicatorem $\frac{1}{S^m}\,\Phi : Sy$; est enim haec formula $B\,(y\,\partial S + S\,\partial y)\,\Phi : Sy$ per se integrabilis. Vnde pro hac parte multiplicator erit $S^{\lambda-m} y^\lambda (1+Sy)^\mu$, qui vtique continet

tinet assumtum $\frac{y^m}{(1+Sy)^n}$, si quidem capiatur $\lambda = m$ et $\mu = -n$. Est vero

$$\int \frac{y^m}{(1+Sy)^n} . B S^m (y \partial S + S \partial y) = B \int \frac{v^m \partial v}{(1+v)^n},$$

posito $Sy = v$.

Corollarium 3.

505. Pro altera parte, quae posito $S = \frac{1}{v}$ abit in

$$\frac{C}{v^n} [-(n-1) y \partial v + (m+1) v \partial y + (m+2-n) y \partial y],$$

habebimus

$$-\frac{(n-1) C y}{v^n} \left(\partial v - \frac{(m+1) v \partial y}{(n-1) y} - \frac{(m+2-n) \partial y}{(n-1)}\right) =$$

$$-\frac{(n-1) C y^{\frac{m+n}{n-1}}}{v^n} \left(y^{\frac{-m-1}{n-1}} \partial v - \frac{m+1}{n-1} y^{\frac{-m-n}{n-1}} v \partial y - \frac{m+2-n}{n-1} y^{\frac{-m-1}{n-1}} \partial y\right)$$

$$= -\frac{(n-1) C y^{\frac{m+n}{n-1}}}{v^n} \partial . \left(y^{\frac{-m-n}{n-1}} v + y^{\frac{n-m-1}{n-1}}\right).$$

Ideoque haec altera pars ita repraesentabitur

$$-(n-1) C S^n y^{\frac{m+n}{n-1}} \partial . \frac{1+Sy}{y^{\frac{m+1}{n-1}} S}.$$

Multiplicator ergo hanc partem integrabilem reddens erit in genere

$$\frac{1}{S^n y^{\frac{m+n}{n-1}}} \Phi : \frac{1+Sy}{S y^{\frac{m+1}{n-1}}}.$$

Corol-

Corollarium 4.

506. Pro altera ergo parte multiplicator erit $\frac{(1+Sy)^{\mu}}{S^{n+\mu}y^{\frac{m+n+\mu(m+1)}{n-1}}}$, quo haec pars fit

$$-(n-1)C.\frac{(1+Sy)^{\mu}}{S^{\mu}y^{\frac{\mu(m+1)}{n-1}}}\partial.\frac{1+Sy}{y^{\frac{m+1}{n-1}}S},$$

cuius integrale est

$$-\frac{(n-1)Cz^{\mu+1}}{\mu+1}, \text{ posito } z=\frac{1+Sy}{y^{\frac{m+1}{n-1}}S}.$$

Corollarium 5.

507. Iam multiplicator pro prima parte

$$S^{\lambda-m}y^{\lambda}(1+Sy)^{\mu}$$

congruens reddetur cum multiplicatore alterius partis modo exhibito, si sumatur $\lambda=m$ et $\mu=-n$, vnde resultat multiplicator communis $\frac{y^m}{(1+Sy)^n}$, hincque posito $Sy=v$ et $\frac{1+Sy}{y^{\frac{m+1}{n-1}}S}=z$, nostrae aequationis integrale erit

$$B\int\frac{v^m\partial v}{(1+v)^n}+Cz^{1-n}=D \text{ siue}$$

$$B\int\frac{v^m\partial v}{(1+v)^n}+\frac{CS^{n-1}y^{m+1}}{(1+Sy)^{n-1}}=D.$$

Scholion.

508. Aequatio ergo, quam hoc problemate integrare didicimus, per principia iam supra stabilita tractari potest, dum

 pro

pro binis eius partibus seorsim multiplicatores quaeruntur, iique inter se congruentes redduntur, cuius methodi hic insignem vsum declarauimus. Possemus etiam multiplicatori hanc formam dare $\frac{y^m}{(1+Sy+Tyy)^n}$, ita vt haec aequatio

$$\frac{y^m[yP\partial x+(Qy+R)\partial y]}{(1+Sy+Tyy)^n}=0$$

per se debeat esse integrabilis, et calculo vt ante instituto inveniemus

$$y^m\left\{\begin{matrix}+(m+1)P\partial x\\-\partial R\end{matrix}\right\}+y^{m+1}\left\{\begin{matrix}+(m+1-n)PS\partial x\\-\partial Q\\-S\partial R\\+nR\partial S\end{matrix}\right\}+y^{m+2}\left\{\begin{matrix}+(m+1-n)PT\partial x\\-S\partial Q\\-T\partial R\\+nQ\partial S\\+nR\partial T\end{matrix}\right\}$$

$$+y^{m+3}\left\{\begin{matrix}-T\partial Q\\+nQ\partial T\end{matrix}\right\}=0,$$

Vnde ex vltimo membro $-T\partial Q+nQ\partial T=0$ concludimus $Q=AT^n$, et ex primo $P\partial x=\frac{\partial R}{m+1}$, qui valores in binis mediis substituti praebent

$$R\partial S-\tfrac{S\partial R}{m+1}-AT^{n-1}\partial T=0 \text{ et}$$

$$R\partial T-\tfrac{2T\partial R}{m+1}+AT^n\partial S-AST^{n-1}\partial T=0,$$

quarum illa fit integrabilis per se si $m=-2$, haec vero integrari potest si $m=2n-1$, fit enim

$$R\partial T-\tfrac{T\partial R}{n}+AT^{n-1}(T\partial S-S\partial T)=0, \text{ seu}$$

$$\frac{nR\partial T-T\partial R}{nT^{n+1}}+\frac{A(T\partial S-S\partial T)}{TT}=0,$$

cuius integrale est $\frac{-R}{nT^n}+\frac{AS}{T}=\frac{-B}{n}$; hincque

R =

$$R = B T^n + n A T^{n-1} S.$$

Praeterea vero notari meretur casus $m = -1$, quem cum illis in subiunctis exemplis euoluamus.

Exemplum 1.

509. *Definire hanc aequationem*

$$y P \partial x + (Q y + R) \partial y = 0,$$

vt multiplicata per $\frac{1}{y(1 + S y + T y y)}$ *fiat per se integrabilis.*

Ob $m = -1$, habemus statim $\partial R = 0$, ideoque $R = C$: tum est vt ante $Q = A T^n$ et $\partial Q = n A T^{n-1} \partial T$, vnde binae reliquae determinationes erunt:

$$- P S \partial x - A T^{n-1} \partial T + C \partial S = 0$$
$$- 2 P T \partial x - A S T^{n-1} \partial T + A T^n \partial S + C \partial T = 0,$$

hinc eliminando $P \partial x$ prodit

$$A S S T^{n-1} \partial T - 2 A T^n \partial T - A T^n S \partial S$$
$$+ 2 C T \partial S - C S \partial T = 0.$$

Statuatur hic $S S = T v$, vt fiat

$$2 T \partial S - S \partial T = T S \left(\tfrac{2 \partial S}{S} - \tfrac{\partial T}{T}\right) = \tfrac{T S \partial v}{v} = \tfrac{T \partial v \surd T}{\surd v},$$

eritque

$$\tfrac{1}{2} A T^n v \partial T - 2 A T^n \partial T - \tfrac{1}{2} A T^{n+1} \partial v + \tfrac{C T \partial v \surd T}{\surd v} = 0,$$

seu hoc modo

$$- \tfrac{1}{2} A T^{n+2} \partial . \tfrac{v - 4}{T} + \tfrac{C T \partial v \surd T}{\surd v} = 0,$$

cuius prior pars integrabilis redditur per multiplicatorem

$$\frac{1}{T^{n+2}} \Phi : \frac{v - 4}{T},$$

posterior vero per $\frac{1}{T\sqrt{T}}\Phi:v$, vnde communis multiplicator erit $\frac{1}{T(v-4)^{n+\frac{1}{2}}\sqrt{T}}$, hincque aequatio elicitur integralis haec

$$\frac{A\,T^{n-\frac{1}{2}}}{(2n-1)(v-4)^{n-\frac{1}{2}}}+C\int\frac{\partial v}{(v-4)^{n+\frac{1}{2}}\sqrt{v}}=D,$$

vnde T definitur per v; tum vero est $S=\sqrt{T\,v}$, $R=C$,

$$Q=A\,T^n,\text{ et }P\partial x=\frac{C\,\partial S-A\,T^{n-1}\,\partial T}{S}.$$

Corollarium 1.

510. Casu quo est $n=\frac{1}{2}$, ob $\frac{1}{0}z^0=lz$, habetur

$\frac{1}{2}A\,l\frac{T}{v-4}+C\int\frac{\partial v}{(v-4)\sqrt{v}}=\frac{1}{2}D$, seu

$\frac{1}{2}A\,l\frac{T}{v-4}-\frac{1}{2}C\,l\frac{\sqrt{v}+2}{\sqrt{v}-2}=\frac{1}{2}D$:

vnde posito $v=4uu$ et $C=\lambda A$, erit

$l\frac{T}{1-uu}-\lambda\,l\frac{1+u}{1-u}=$ Const. seu

$T=E(1-uu)\left(\frac{1+u}{1-u}\right)^\lambda$. Hinc porro

$S=2u\sqrt{T}=2u\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}$, et

$R=C=\lambda A$; tum $Q=A\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}$, atque

$P\partial x=\frac{\lambda A\partial u}{u}+\frac{\lambda A\partial T}{2T}-\frac{A\partial T}{2Tu}$.

At est $\frac{\partial T}{T}=\frac{-2u\partial u+2\lambda\partial u}{1-uu}$. Ergo $P\partial x=\frac{A\partial u(1+\lambda\lambda-2\lambda u)}{1-uu}$.

Quocirca pro hac aequatione

$$\frac{A\,y\,\partial u(1+\lambda\lambda-2\lambda u)}{1-uu}+A\,\partial y\left[\lambda+y\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}\right]=0$$

multi-

multiplicator erit

$$\frac{1}{y\sqrt{[1+2uy\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}+Eyy(1-uu)\left(\frac{1+u}{1-u}\right)^{\lambda}]}}.$$

Corollarium 2.

511. Casu quo $n=-\frac{1}{2}$ habemus

$$-\frac{A(v-4)}{2T}+2C\sqrt{v}=-2D, \text{ seu } T=\frac{A(v-4)}{4D+4C\sqrt{v}}.$$

Ponamus $v=4uu$, vt sit $T=\frac{A(uu-1)}{D+2Cu}$, tum fit

$$S=2u\sqrt{T}=2u\sqrt{\frac{A(uu-1)}{D+2Cu}},$$

$$R=C,\ Q=\sqrt{\frac{A(D+2Cu)}{uu-1}}, \text{ et}$$

$$P\partial x=\frac{C\partial u}{u}+\frac{C\partial T}{2T}-\frac{A\partial T}{2TTu}=\frac{\partial u(C+Du+Cuu)(Cu^3-3Cu-D)}{u(uu-1)^2(D+2Cu)},$$

vnde tam aequatio quam multiplicator definitur.

Exemplum 2.

512. *Definire aequationem*

$$yP\partial x+(Qy+R)\partial y=0,$$

vt multiplicata per $\frac{1}{y^2(1+Sy+Tyy)^n}$, *fiat per se integrabilis.*

Ob $m=-2$, ex superioribus habemus:

$$RS=\frac{A}{n}T^n+B, \text{ seu } R=\frac{AT^n}{nS}+\frac{B}{S},$$

qui valor in altera aequatione substitutus praebet

$$\frac{(2n+1)AT^n\partial T}{nS}-\frac{2AT^{n+1}\partial S}{nSS}+AT^n\partial S-AST^{n-1}\partial T$$
$$+\frac{B\partial T}{S}-\frac{2BT\partial S}{SS}=0,$$

quae

quae in has tres partes diſtinguatur

$$\frac{AS}{nT^n}\left(\frac{(2n+1)T^{2n}\partial T}{S^2}-\frac{2T^{2n+1}\partial S}{S^3}\right)+AT^{n+1}\left(\frac{\partial S}{T}-\frac{S\partial T}{TT}\right)$$

$$+BS\left(\frac{\partial T}{SS}-\frac{2BT\partial S}{S^3}\right)=0, \text{ ſeu}$$

$$\frac{AS}{nT^n}\partial.\frac{T^{2n+1}}{SS}+AT^{n+1}\partial.\tfrac{S}{T}+BS\partial.\tfrac{T}{SS}=0.$$

Statuamus ad abreuiandum

$$\frac{T^{2n+1}}{SS}=p,\ \tfrac{S}{T}=q \text{ et } \tfrac{T}{SS}=r,$$

fiet $S=\frac{1}{qr}$, $T=\frac{1}{qqr}$, hinc $p=\frac{1}{q^{4n}r^{2n-1}}$; noſtraque aequatio ita ſe habebit

$$\frac{A}{nq\sqrt{pr}}\partial p+\frac{A\sqrt{p}}{qqr\sqrt{r}}\partial q+\frac{B}{qr}\partial r=0, \text{ ſeu}$$

$$\frac{A\sqrt{r}}{n\sqrt{p}}\partial p+\frac{A\sqrt{p}}{q\sqrt{r}}\partial q+B\partial r=0.$$

Quas tres partes ſeorſim conſideremus, ac prima ſit integrabilis multiplicata per $\frac{\sqrt{p}}{\sqrt{r}}\Phi:p$, ſecunda vero per $\frac{q\sqrt{r}}{\sqrt{r}}\Phi:q$, tertia vero per $\Phi:r$. Vt bini primi conueniant, ponatur

$$\frac{\sqrt{p}}{\sqrt{r}}\cdot p^\lambda=\frac{q\sqrt{r}}{\sqrt{p}}\cdot q^\mu \text{ ſeu } p^{\lambda+1}=q^{\mu+1}r, \text{ hinc}$$

$$p=q^{\frac{\mu+1}{\lambda+1}}r^{\frac{1}{\lambda+1}}=q^{-4n}r^{-2n+1}.$$

Fit ergo

$$\lambda+1=-\frac{1}{2n-1} \text{ et } \mu+1=-4n(\lambda+1)=\frac{4n}{2n-1}, \text{ ſicque}$$

$$\mu=\frac{2n+1}{2n-1} \text{ et } \lambda=-\frac{2n}{2n+1}.$$

Multiplicetur ergo aequatio per $\dfrac{q^{\frac{4n}{2n-1}}\sqrt{r}}{\sqrt{p}}=q^{2n+\frac{4n}{2n-1}}r^{n+1}$,

ac

ac prodibit

$$\tfrac{A}{n} p^{\lambda}\, \partial p + A q^{\mu}\, \partial q + B q^{2n + \frac{4n}{2n+1}} r^{n+1}\, \partial r = 0,$$

seu

$$A\, \partial . \left(\frac{p^{\lambda+1}}{n(\lambda+1)} + \frac{q^{\mu+1}}{\mu+1}\right) + B q^{\frac{4nn+6n}{2n+1}} r^{n+1}\, \partial r = 0,$$

vel

$$\frac{(2n+1)A}{4n}\, \partial . q^{\frac{4n}{2n+1}} (1-4r) + B q^{\frac{4nn+6n}{2n+1}} r^{n+1}\, \partial r = 0.$$

Multiplicetur per $q^{\frac{4\nu n}{2n+1}} (1-4r)^{\nu}$, vt prodeat

$$\frac{(2n+1)A}{4n} . q^{\frac{4\nu n}{2n+1}} (1-4r)^{\nu}\, \partial . q^{\frac{4n}{2n+1}} (1-4r)$$
$$+ B q^{\frac{4nn+6n+4\nu n}{2n+1}} r^{n+1}\, \partial r\, (1-4r)^{\nu} = 0.$$

Fiat ergo $4\nu + 4n + 6 = 0$ seu $\nu = -n - \frac{3}{2}$, et ambo membra integrari poterunt, eritque

$$\frac{(2n+1)A}{4n(\nu+1)} q^{\frac{4n(\nu+1)}{2n+1}} (1-4r)^{\nu+1} + B\int r^{n+1} \partial r\, (1-4r)^{\nu} = \text{Conſt.}$$

at eſt $\nu + 1 = -n - \frac{1}{2} = -\frac{2n-1}{2}$, ſicque habebitur

$$-\frac{A}{2n} q^{-2n} (1-4r)^{\frac{-2n-1}{2}} + B\int \frac{r^{n+1}\, \partial r}{(1-4r)^{\frac{2n+3}{2}}} = \text{Conſt.}$$

Dabitur ergo q per r, eritque $S = \frac{1}{qr}$, $T = \frac{s}{q}$, tum $R = \frac{A\,T^n}{n\,S} + \frac{B}{S}$, $Q = A\,T^n$ et $P\,\partial x = -\partial R$.

Corollarium 1.

513. Si ſit $n = -\frac{1}{2}$, erit $A\,q + \frac{2Br\sqrt{r}}{3} = \frac{C}{3}$, ſeu $q = \frac{C - 2Br\sqrt{r}}{3A}$; hincque
$S = \frac{3A}{Cr - 2Br^2\sqrt{r}}$, $T = \frac{9AA}{r(C - 2Br\sqrt{r})^2}$, $Q = \frac{C\sqrt{r} - 2Brr}{3}$ et

 $R =$

$$R = \frac{Q + nB}{rS} = \frac{B - 2Q}{S} = \frac{r(C - 2Br\sqrt{r})(3B - 2C\sqrt{r} + 4Brr)}{9A} \text{ seu}$$

$$R = \frac{3BCr - 2CCr\sqrt{r} - 6BBrr\sqrt{r} + 8BCr^3 - 8BBr^4\sqrt{r}}{9A}.$$

Corollarium 2.

514. Ponamus eodem casu $r = uu$, erit

$$S = \frac{3A}{Cuu - 2Bu^5}, \quad T = \frac{9AA}{uu(C - 2Bu^3)^2}, \quad Q = \frac{u(C - 2Bu^3)}{3}, \text{ et}$$

$$R = \frac{3BCu^2 - 2CCu^3 - 6BBu^5 + 8BCu^6 - 8BBu^9}{9A}, \text{ hincque}$$

$$P\,\partial x = \frac{-6BCu + 6CCuu + 30BBu^4 - 48BCu^5 + 72BBu^8}{9A}\,\partial u,$$

eritque aequatio $y P\,\partial x + (Qy + R)\,\partial y = 0$ integrabilis, si multiplicetur per

$$\frac{\sqrt{(1 + Sy + Tyy)}}{yy} = \frac{1}{yy}\sqrt{\left(1 + \frac{3Ay}{uu(C - 2Bu^3)} + \frac{9AAyy}{uu(C - 2Bu^3)^2}\right)}.$$

Exemplum 3.

515. *Definire aequationem*

$$y P\,\partial x + (Qy + R)\,\partial y = 0,$$

quae multiplicata per $\dfrac{y^{2n-1}}{(1 + Sy + Tyy)^n}$ *fiat per se integrabilis.*

Hic est $m = 2n - 1$, $Q = AT^n$, et $P\,\partial x = \frac{\partial R}{2n}$; tum vero ex superioribus $R = nAT^{n-1}S + BT^n$, ac superest aequatio

$$R\,\partial S - \frac{S\,\partial R}{2n} - AT^{n-1}\,\partial T = 0,$$

quae loco R substituto valore inuento, abit in

$$(2n-1)AT^{n-1}S\,\partial S - (n-1)AT^{n-2}SS\,\partial T - 2AT^{n-1}\partial T$$
$$+ 2BT^n\,\partial S - BT^{n-1}S\,\partial T = 0, \text{ seu}$$

$$(2n-1)ATS\,\partial S - (n-1)ASS\,\partial T - 2AT\,\partial T$$
$$+ 2BTT\,\partial S - BTS\,\partial T = 0.$$

Prius

Prius membrum posito $SS = u$ abit in

$$(n-\tfrac{1}{2})AT\partial u - (n-1)Au\partial T - 2AT\partial T, \text{ seu}$$

$$(n-\tfrac{1}{2})AT\left(\partial u - \frac{(n-1)u\partial T}{(n-\frac{1}{2})T} - \frac{2\partial T}{n-\frac{1}{2}}\right), \text{ siue}$$

$$\tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}\left(\frac{\partial u}{T^{\frac{2n-2}{2n-1}}} - \frac{2(n-1)u\partial T}{(2n-1)T^{\frac{4n-3}{2n-1}}} - \frac{4\partial T}{(2n-1)T^{\frac{2n-2}{2n-1}}}\right)$$

$$=(2n-1)AT^{\frac{4n-3}{2n-1}}\partial.\left(\frac{u}{T^{\frac{2n-2}{2n-1}}} - 4T^{\frac{1}{2n-1}}\right), \text{ vel}$$

$$\tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}\partial.T^{\frac{1}{2n-1}}\left(\tfrac{SS}{T}-4\right)+\tfrac{BT^2}{S}\partial.\tfrac{SS}{T}=0, \text{ seu}$$

$$(2n-1)AT^{\frac{-1}{2n-1}}\partial.T^{\frac{1}{2n-1}}\left(\tfrac{SS}{T}-4\right)+\tfrac{2BT}{S}\partial.\tfrac{SS}{T}=0.$$

Ponatur $\frac{SS}{T} = p$ et

$$T^{\frac{1}{2n-1}}\left(\tfrac{SS}{T}-4\right) = q = T^{\frac{1}{2n-1}}(p-4),$$

vt sit $T^{\frac{1}{2n-1}} = \frac{q}{p-4}$, vnde

$$T = \frac{q^{2n-1}}{(p-4)^{2n-1}} \text{ et } S = \sqrt{\frac{pq^{2n-1}}{(p-4)^{2n-1}}}.$$

Ergo

$$\frac{(2n-1)A(p-4)\partial q}{q} + \frac{2B\sqrt{q^{2n-1}}}{\sqrt{p(p-4)^{2n-1}}}\partial p = 0$$

siue

$$\frac{(2n-1)A\partial q}{q^{n+\frac{1}{2}}} + \frac{2B\partial p:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}} = 0,$$

quae integrata praebet

$$\frac{-2A}{q^{n-\frac{1}{2}}} + 2B\int\frac{\partial p:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}} = 2C,$$

 et

et facto $\frac{p}{p-4} = vv$, seu $p = \frac{4vv}{vv-1}$, fiet

$$\frac{+A}{q^{n-\frac{1}{2}}} - \frac{B}{4^{n-1}} \int \partial v (vv-1)^{n-1} = C.$$

Scholion.

516. Haec fusius non prosequor, quia ista exempla eum in finem potissimum attuli, vt methodus supra tradita aequationes differentiales tractandi exerceretur; in his enim exemplis casus non parum difficiles se obtulerunt, quos ita per partes resoluere licuit, vt pro singulis multiplicatores idonei quaererentur, ex iisque multiplicator communis definiretur; nunc igitur alia aequationum genera, quae per multiplicatores integrabiles reddi queant, inuestigemus.

Problema 67.

517. Ipsius x functiones P, Q, R, S definire, vt haec aequatio $(Py+Q)\partial x + y\partial y = 0$, per hunc multiplicatorem $(yy+Ry+S)^n$ integrabilis reddatur.

Solutio.

Necesse igitur est, sit

$$\left(\frac{\partial.(Py+Q)(yy+Ry+S)^n}{\partial y}\right) = \left(\frac{\partial.y(yy+Ry+S)^n}{\partial x}\right)$$

vnde colligitur per $(yy+Ry+S)^{n-1}$ diuidendo

$$P(yy+Ry+S)+n(Py+Q)(2y+R) = \frac{ny(y\partial R+\partial S)}{\partial x}$$

seu

$$\left.\begin{array}{lll}(2n+1)Pyy\partial x & +(n+1)PRy\partial x & +PS\partial x \\ -nyy\partial R & +2nQy\partial x & +nQR\partial x \\ & -ny\partial S & \end{array}\right\} = 0.$$

Hinc

Hinc ergo concluditur $P\,\partial x = \frac{n\,\partial R}{2n+1}$, et

$$\frac{(n+1)R\,\partial R}{2n+1} + 2\,Q\,\partial x - \partial S = 0,$$

$$\frac{S\,\partial R}{2n+1} + Q\,R\,\partial x = 0, \text{ porroque}$$

$$Q\,\partial x = \frac{-S\,\partial R}{(2n+1)R} = \frac{-(n+1)R\,\partial R}{2(2n+1)} + \frac{\partial S}{2}: \text{ ergo}$$

$$\partial S + \frac{2\,\partial R}{(2n+1)R} = \frac{(n+1)R\,\partial R}{2n+1},$$

quae per $R^{\frac{2}{2n+1}}$ multiplicata et integrata, dat

$$R^{\frac{2}{2n+1}}\,S = C + \tfrac{1}{4}\,R^{\frac{4n+4}{2n+1}}, \text{ hincque}$$

$$S = \tfrac{1}{4}\,R\,R + C\,R^{\frac{-2}{2n+1}}, \text{ atque}$$

$$Q\,\partial x = \frac{-R\,\partial R}{4(2n+1)} - \frac{C}{2n+1}\,R^{\frac{-2n-3}{2n+1}}\,\partial R, \text{ et } P\,\partial x = \frac{n\,\partial R}{2n+1}:$$

vnde aequationem obtinemus

$$(n\,y - \tfrac{1}{4}R - C\,R^{\frac{-2n-3}{2n+1}})\,\partial R + (2\,n+1)\,y\,\partial y = 0,$$

quae integrabilis redditur per hunc multiplicatorem

$$(y\,y + R\,y + \tfrac{1}{4}R\,R + C\,R^{\frac{-2}{2n+1}})^n.$$

Corollarium 1.

518. Casu quo $n = -\frac{1}{2}$, fit $\partial R = 0$ et $R = A$, et reliquae aequationes sunt,

$$(n+1)\,A\,P\,\partial x + 2\,n\,Q\,\partial x - n\,\partial S = 0 \text{ et}$$

$$P\,S\,\partial x + n\,A\,Q\,\partial x = 0.$$

Ergo $P\,\partial x = \frac{A\,Q\,\partial x}{2S} = \frac{2\,Q\,\partial x - \partial S}{A}$, ideoque

$$(A\,A - 4\,S)\,Q\,\partial x = -2\,S\,\partial S, \text{ seu}$$

$$Q\,\partial x = \frac{-2\,S\,\partial S}{A\,A - 4\,S} \text{ et } P\,\partial x = \frac{-A\,\partial S}{A\,A - 4\,S}$$

sicque haec aequatio $\frac{(Ay+2S)\partial S}{4S-AA}+y\,\partial y=0$ integrabilis redditur per hunc multiplicatorem $\frac{1}{\sqrt{(yy+Ay+S)}}$.

Corollarium 2.

519. Si hic ponamus $A=2a$ et $S=x$, haec aequatio $\frac{(ay+x)\partial x+2y\partial y(x-aa)}{(x-aa)\sqrt{(yy+2ay+x)}}=0$ per se est integrabilis, vnde integrale inueniri potest huius aequationis

$$x\,\partial x+ay\,\partial x+2\,xy\,\partial y-2\,aay\,\partial y=0,$$

quae diuisa per $(x-aa)\sqrt{(yy+2ay+x)}$ fit integrabilis.

Corollarium 3.

520. Ad integrale inueniendum, sumatur primo x constans, et partis $\frac{2y\partial y}{\sqrt{(yy+2ay+x)}}$ integrale est

$$2\sqrt{(yy+2ay+x)}+2al[a+y-\sqrt{(yy+2ay+x)}]+X,$$

cuius differentiale sumto y constante

$$\frac{\partial x}{\sqrt{(yy+2ay+x)}}-\frac{a\partial x:\sqrt{(yy+2ay+x)}}{a+y-\sqrt{(yy+2ay+x)}}+\partial X,$$

si alteri aequationis parti $\frac{(ay+x)\partial x}{(x-aa)\sqrt{(yy+2ay+x)}}$ aequetur, reperitur $\partial X=\frac{a\partial x}{aa-x}$ et $X=-a\,l\,(aa-x)$. Ex quo integrale completum erit

$$\sqrt{(yy+2ay+x)}+a\,l\,\frac{a+y-\sqrt{(yy+2ay+x)}}{\sqrt{(aa-x)}}=C.$$

Corollarium 4.

521. Memoratu dignus est etiam casus $n=-1$, qui scripto a loco $C+\frac{1}{4}$ praebet hanc aequationem

$$(y+aR)\,\partial R+y\,\partial y=0,$$

quae diuisa per $yy+Ry+aRR$ fit integrabilis, haec autem aequatio est homogenea.

Scho-

Scholion.

522. Poteſt etiam aequationis

$$(Py+Q)\partial x+y\partial y=0$$

multiplicator ſtatui $(y+R)^m(y+S)^n$, fierique debet

$$\left(\frac{\partial.(Py+Q)(y+R)^m(y+S)^n}{\partial y}\right)=\left(\frac{\partial.y(y+R)^m(y+S)^n}{\partial x}\right);$$

vnde reperitur

$$\begin{aligned}&P\partial x(y+R)(y+S)+m\partial x(Py+Q)(y+S)\\&\quad+n\partial x(Py+Q)(y+R)=my(y+S)\partial R+ny(y+R)\partial S,\end{aligned}$$

quae euoluitur in

$$\left.\begin{array}{lll}(m+n+1)Pyy\partial x&+(n+1)PRy\partial x&+PRS\partial x\\-myy\partial R&+(m+1)PSy\partial x&+mQS\partial x\\-nyy\partial S&+(m+n)Qy\partial x&+nQR\partial x\\&-mSy\partial R&\\&-nRy\partial S&\end{array}\right\}=0$$

vnde colligitur

$$P\partial x=\frac{m\partial R+n\partial S}{m+n+1}\ \text{et}\ Q\partial x=\frac{-PRS\partial x}{mS+nR}=\frac{-RS(m\partial R+n\partial S)}{(m+n+1)(mS+nR)},$$

hincque

$$\frac{(m\partial R+n\partial S)[(n+1)R+(m+1)S]}{m+n+1}-\frac{(m+n)RS(m\partial R+n\partial S)}{(m+n+1)(mS+nR)}-mS\partial R-nR\partial S=0,$$

ſeu

$$\left.\begin{array}{l}+m(n+1)R\partial R-mnR\partial S\\+n(m+1)S\partial S-mnS\partial R\end{array}\right.-\frac{m(m+n)RS\partial R-n(m+n)RS\partial S}{mS+nR}=0,$$

quae reducitur ad hanc formam

$$\left.\begin{array}{l}+(n+1)RR\partial R+(m-n-1)RS\partial R-SS\partial R\\+(m+1)SS\partial S+(n-m-1)RS\partial S-RR\partial S\end{array}\right\}=0,$$

quae cum ſit homogenea, diuidatur per

$$(n+1)R^3+(m-2n-1)R^2S+(n-2m-1)RSS+(m+1)S^3,$$

ſeu

feu per

$$(R-S)^{2}\,[(n+1)\,R+(m+1)\,S]$$

vt fiat integrabilis. At ipfa illa aequatio per $R-S$ diuifa, erit

$$(n+1)R\,\partial R+mS\,\partial R-nR\,\partial S-(m+1)\,S\,\partial S=0.$$

Diuidatur per

$$(R-S)\,[(n+1)\,R+(m+1)\,S]$$

et refoluatur in fractiones partiales, erit

$$\frac{\partial R}{m+n+2}\left(\frac{m+n+1}{R-S}+\frac{n+1}{(n+1)R+(m+1)S}\right)$$
$$+\frac{\partial S}{m+n+2}\left(\frac{m+n+1}{S-R}+\frac{m+1}{(n+1)R+(m+1)S}\right)=0$$

feu

$$\frac{(m+n+1)(\partial R-\partial S)}{R-S}+\frac{(n+1)\,\partial R+(m+1)\,\partial S}{(n+1)R+(m+1)S}=0;$$

vnde integrando obtinemus,

$$(R-S)^{m+n+1}\,[(n+1)\,R+(m+1)\,S]=C.$$

Sit $R-S=u$, erit

$$(n+1)\,R+(m+1)\,S=\frac{C}{u^{m+n+1}},$$

hincque

$$R=\frac{(m+1)\,u}{m+n+2}+\frac{a}{u^{m+n+1}},\ \text{et}$$

$$S=\frac{-(n+1)\,u}{m+n+2}+\frac{a}{u^{m+n+1}},$$

tum vero

$$P\,\partial x=\frac{(m-n)\,\partial u}{m+n+2}-\frac{(m+n)\,a\,\partial u}{u^{m+n+2}},\ \text{et}$$

$$Q\,\partial x=\frac{\partial u}{u}\left(\frac{a}{u^{m+n+1}}+\frac{(m+1)\,u}{m+n+2}\right)\left(\frac{a}{u^{m+n+1}}-\frac{(n+1)\,u}{m+n+2}\right).$$

Corol-

Corollarium 1.

523. Hinc ergo integrari potest ista aequatio

$$y\,\partial y + y\,\partial u\left(\frac{m-n}{m+n+2} - \frac{(m+n)\,a}{u^{m+n+2}}\right)$$

$$+\frac{\partial u}{u}\left(\frac{a\,a}{u^{2m+2n+2}} + \frac{(m-n)\,a}{(m+n+2)\,u^{m+n}} - \frac{(m+1)(n+1)\,u\,u}{(m+n+2)^2}\right) = 0,$$

quippe quae per se sit integrabilis, si multiplicetur per

$$\left(y + \frac{a}{u^{m+n+1}} + \frac{(m+1)\,u}{m+n+2}\right)^m \left(y + \frac{a}{u^{m+n+1}} - \frac{(n+1)\,u}{m+n+2}\right)^n.$$

Corollarium 2.

524. Sit $m = n$, et aequatio nostra erit

$$y\,\partial y - \frac{2\,n\,a\,y\,\partial u}{u^{2n+2}} + \frac{a\,a\,\partial u}{u^{4n+3}} - \tfrac{1}{4}\,u\,\partial u = 0,$$

cuius multiplicator est $[(y + \frac{a}{u^{2n+1}})^2 - \frac{1}{4}\,u\,u]^n$. Quare si ponamus $y = z - \frac{a}{u^{2n+1}}$, aequatio prodit

$$z\,\partial z - \frac{a\,\partial z}{u^{2n+1}} + \frac{a\,z\,\partial u}{u^{2n+2}} - \tfrac{1}{4}\,u\,\partial u = 0,$$

quae integrabilis fit multiplicata per $(z\,z - \frac{1}{4}\,u\,u)^n$. Vel ponatur $z = \frac{1}{2}\,y$ et $a = \frac{1}{2}\,b$, erit aequatio

$$y\,\partial y - u\,\partial u - \frac{b\,\partial y}{u^{2n+1}} + \frac{b\,y\,\partial u}{u^{2n+2}} = 0,$$

et multiplicator $(y\,y - u\,u)^n$.

Corollarium 3.

525. Si $m = -n$, prodit haec aequatio

$$y\,\partial y - n\,y\,\partial u + \tfrac{a\,a\,\partial u}{u^3} + \tfrac{1}{4}(n\,n - 1)\,u\,\partial u - \tfrac{n\,a\,\partial u}{u} = 0,$$

 quae

quae integrabilis redditur multiplicata per

$$[y + \tfrac{a}{u} - \tfrac{1}{2}(n+1)u]^n\,[y + \tfrac{a}{u} - \tfrac{1}{2}(n-1)u]^{-n}.$$

Posito autem $y + \frac{a}{u} = z$, prodit haec aequatio

$$z\,\partial z - n z\,\partial u + \tfrac{1}{4}(nn-1)u\,\partial u - \tfrac{a\,\partial z}{u} + \tfrac{a z\,\partial u}{uu} = 0,$$

quam integrabilem reddit hic multiplicator

$$[z - \tfrac{1}{2}(n+1)u]^n\,[z - \tfrac{1}{2}(n-1)u]^{-n}.$$

Corollarium 4.

526. Ponamus hic $z = uv$, et habebitur ista aequatio

$$uuv\,\partial v + u\,\partial u\,[vv - nv + \tfrac{1}{4}(nn-1)] = a\,\partial v,$$

quae si multiplicetur per $\left(\frac{v - \frac{1}{2}(n+1)}{v - \frac{1}{2}(n-1)}\right)^n$, vtrumque membrum fiet integrabile. Posito enim $\frac{v - \frac{1}{2}(n+1)}{v - \frac{1}{2}(n-1)} = s$, seu

$$v = \frac{n+1-(n-1)s}{2(1-s)},$$

oritur

$$\frac{s^{n+1}u\,\partial u}{(1-s)^2} + \frac{n+1-(n-1)s}{2(1-s)^3}\,uus^n\,\partial s = \frac{a\,s^n\,\partial s}{(1-s)^2};$$

cuius integrale est

$$\frac{s^{n+1}uu}{2(1-s)^2} = a\int\frac{s^n\,\partial s}{(1-s)^2}.$$

Scholion.

527. Quo nostram aequationem in genere concinniorem reddamus, ponamus $m = -\lambda - 1 + \mu$ et $n = -\lambda - 1 - \mu$, vt sit $m + n + 2 = -2\lambda$, fietque aequatio

$$y\,\partial y - y\,\partial u\,[\tfrac{\mu}{\lambda} - 2(\lambda+1)a u^{2\lambda}]$$
$$+ u\,\partial u\,(\tfrac{\mu\mu - \lambda\lambda}{4\lambda\lambda} - \tfrac{\mu}{\lambda}a u^{2\lambda} + aa u^{4\lambda}) = 0,$$

quae

quae per hunc multiplicatorem integrabilis redditur

$$(y+au^{2\lambda+1}-\tfrac{(\mu-\lambda)u}{2\lambda})^{\mu-\lambda-1}(y+au^{2\lambda+1}-\tfrac{(\mu+\lambda)u}{2\lambda})^{-\mu-\lambda-1}.$$

Ponatur $y+au^{2\lambda+1}=uz$, et orietur haec aequatio

$$uz\partial z-au^{2\lambda+1}\partial z+\partial u(zz-\tfrac{\mu}{\lambda}z+\tfrac{\mu\mu-\lambda\lambda}{4\lambda\lambda})=0,$$

cui respondet multiplicator

$$u^{-2\lambda-1}(z+\tfrac{\lambda-\mu}{2\lambda})^{\mu-\lambda-1}(z-\tfrac{\lambda-\mu}{2\mu})^{-\mu-\lambda-1}.$$

Reperitur autem integrale

$$C=a\int\partial z(z+\tfrac{\lambda-\mu}{2\lambda})^{\mu-\lambda-1}(z-\tfrac{\lambda-\mu}{2\lambda})^{-\mu-\lambda-1}$$

$$+\frac{1}{2\lambda u^{2\lambda}}(z+\tfrac{\lambda-\mu}{2\lambda})^{\mu-\lambda}(z-\tfrac{\lambda-\mu}{2\lambda})^{-\mu-\lambda},$$

quod ergo conuenit huic aequationi differentiali

$$z\partial z+\tfrac{\partial u}{u}(z+\tfrac{\lambda-\mu}{2\lambda})(z-\tfrac{\lambda-\mu}{2\lambda})=au^{2\lambda}\partial z.$$

Problema 68.

528. Ipsius x functiones P, Q, R et X definire, vt haec aequatio $\partial y+yy\partial x+X\partial x=0$ integrabilis reddatur per hunc multiplicatorem $\frac{1}{Pyy+Qy+R}$.

Solutio.

Debet ergo esse

$$\frac{1}{\partial y}\partial.\frac{yy+X}{Pyy+Qy+R}=\frac{1}{\partial x}\partial.\frac{1}{Pyy+Qy+R},$$

hincque

$$2y(Pyy+Qy+R)-(yy+X)(2Py+Q)$$
$$=-\frac{yy\partial P-y\partial Q-\partial R}{\partial x}:$$

ergo fieri debet

$$\left.\begin{array}{lll}+Qyy\partial x & +2Ry\partial x & -QX\partial x\\ +yy\partial P & -2PXy\partial x & +\partial R\\ & +y\partial Q & \end{array}\right\}=0.$$

 Quare

Quare habetur $Q = -\frac{\partial P}{\partial x} = \frac{\partial R}{X \partial x}$, et $X = -\frac{\partial R}{\partial P}$. Sumto ergo ∂x constante est $\partial Q = -\frac{\partial \partial P}{\partial x}$, vnde fieri oportet

$$2R\partial x + \frac{2P\partial R\partial x}{\partial P} - \frac{\partial\partial P}{\partial x} = 0, \text{ seu}$$
$$R\partial P + P\partial R = \frac{\partial P\partial\partial P}{2\partial x^2},$$

cuius integratio praebet $PR = \frac{\partial P^2}{4\partial x^2} + C$, hinc $R = \frac{\partial P^2}{4P\partial x^2} + \frac{C}{P}$, tum

$$Q = -\frac{\partial P}{\partial x}, \text{ et } X = \frac{C}{PP} + \frac{\partial P^2}{4PP\partial x^2} - \frac{\partial\partial P}{2P\partial x^2}.$$

Ponamus $P = SS$, vt S sit functio quaecunque ipsius x, obtinebimusque

$$P = SS,\ Q = -\frac{2S\partial S}{\partial x},\ R = \frac{C}{SS} + \frac{\partial S^2}{\partial x^2}, \text{ et } X = \frac{C}{S^4} - \frac{\partial\partial S}{S\partial x^2},$$

quibus sumtis valoribus, per se integrabilis erit haec aequatio

$$\frac{\partial y + yy\partial x + X\partial x}{Pyy + Qy + R} = 0.$$

Scholion.

529. Haec solutio commodius institui poterit, si multiplicatori tribuatur haec forma $\frac{P}{yy + 2Qy + R}$, vt fieri debeat

$$\frac{1}{\partial y}\partial \cdot \frac{P(yy + X)}{yy + 2Qy + R} = \frac{1}{\partial x}\partial \cdot \frac{P}{yy + 2Qy + R},$$

vnde oritur

$$\left.\begin{array}{lll} 2PQyy\partial x & + 2PRy\partial x & - 2PQX\partial x \\ - yy\partial P & - 2PXy\partial x & - R\partial P \\ & - 2Qy\partial P & + P\partial R \\ & + 2Qy\partial Q & \end{array}\right\} = 0,$$

vbi ex singulis commode definitur $\frac{\partial P}{P}$: scilicet

$$\frac{\partial P}{P} = 2Q\partial x = \frac{R\partial x - X\partial x + \partial Q}{Q} = \frac{\partial R - 2QX\partial x}{R}.$$

Hinc colligitur $2Q(R + X)\partial x = \partial R$, vnde nunc ipsum elementum ∂x definiamus, $\partial x = \frac{\partial R}{2Q(R+X)}$, quo valore substituto adipiscimur

$Q\partial R$

$$\frac{Q\partial R}{R+X} = \frac{(R-X)\partial R}{2Q(R+X)} + \partial Q \text{ seu}$$

$$2QQ\partial R = R\partial R - X\partial R + 2QR\partial Q + 2QX\partial Q:$$

vnde colligimus

$$X = \frac{2QQ\partial R - 2QR\partial Q - R\partial R}{2Q\partial Q - \partial R}, \text{ et } R + X = \frac{2(QQ-R)\partial R}{2Q\partial Q - \partial R},$$

hinc $\partial x = \frac{2Q\partial Q - \partial R}{4Q(QQ-R)}$, atque $\frac{\partial P}{P} = \frac{2Q\partial Q - \partial R}{2(QQ-R)}$; ideoque

$$P = A\sqrt{(QQ-R)}.$$

Fiat $QQ - R = S$, ac reperietur

$$\partial x = \frac{\partial S}{4QS}, \; X = \frac{4QS\partial Q}{\partial S} - QQ - S, \; R = QQ - S,$$

atque $P = A\sqrt{S}$. Quocirca habebimus hanc aequationem

$$\partial y + \frac{yy\partial S}{4QS} + \partial Q - \frac{(QQ+S)\partial S}{4QS} = 0,$$

quae integrabilis redditur per hunc multiplicatorem

$$\frac{\sqrt{S}}{yy+2Qy+QQ-S} = \frac{\sqrt{S}}{(y+Q)^2 - S}.$$

Ad eius integrale inueniendum, sumantur Q et S constantes, prodibitque

$$\int \frac{\partial y\sqrt{S}}{(y+Q)^2 - S} = \frac{1}{2} l \frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}} + V,$$

existente V certa functione ipsius S vel Q. Iam differentietur haec forma sumta y constante, proditque

$$\frac{\partial Q\sqrt{S} - \frac{(Q+y)\partial S}{2\sqrt{S}}}{(y+Q)^2 - S} + \partial V = \frac{yy\partial S + 4QS\partial Q - QQ\partial S - S\partial S}{4Q[(y+Q)^2 - S]\sqrt{S}},$$

ideoque

$$\partial V = \frac{yy\partial S + 2Qy\partial S + QQ\partial S - S\partial S}{4Q[(y+Q)^2 - S]\sqrt{S}} = \frac{\partial S}{4Q\sqrt{S}}.$$

Ex quo aequationis nostrae integrale est

$$\frac{1}{2} l \frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}} + \frac{1}{4}\int \frac{\partial S}{Q\sqrt{S}} = C.$$

 Corol-

Quare habetur $Q = -\frac{\partial P}{\partial x} = \frac{\partial R}{X \partial x}$, et $X = -\frac{\partial R}{\partial P}$. Sumto ergo ∂x constante est $\partial Q = -\frac{\partial \partial P}{\partial x}$, vnde fieri oportet

$$2R\partial x + \frac{2P\partial R\partial x}{\partial P} - \frac{\partial\partial P}{\partial x} = 0, \text{ seu}$$
$$R\partial P + P\partial R = \frac{\partial P\partial\partial P}{2\partial x^2},$$

cuius integratio praebet $PR = \frac{\partial P^2}{4\partial x^2} + C$, hinc $R = \frac{\partial P^2}{4P\partial x^2} + \frac{C}{P}$, tum

$$Q = -\frac{\partial P}{\partial x}, \text{ et } X = \frac{C}{PP} + \frac{\partial P^2}{4PP\partial x^2} - \frac{\partial\partial P}{2P\partial x^2}.$$

Ponamus $P = SS$, vt S sit functio quaecunque ipsius x, obtinebimusque

$$P = SS,\ Q = -\frac{2S\partial S}{\partial x},\ R = \frac{C}{SS} + \frac{\partial S^2}{\partial x^2}, \text{ et } X = \frac{C}{S^4} - \frac{\partial\partial S}{S\partial x^2},$$

quibus sumtis valoribus, per se integrabilis erit haec aequatio

$$\frac{\partial y + yy\partial x + X\partial x}{Pyy + Qy + R} = 0.$$

Scholion.

529. Haec solutio commodius institui poterit, si multiplicatori tribuatur haec forma $\frac{P}{yy + 2Qy + R}$, vt fieri debeat

$$\frac{1}{\partial y}\partial \cdot \frac{P(yy+X)}{yy+2Qy+R} = \frac{1}{\partial x}\partial \cdot \frac{P}{yy+2Qy+R},$$

vnde oritur

$$\left.\begin{array}{lll} 2PQyy\partial x & + 2PRy\partial x & - 2PQX\partial x \\ -yy\partial P & - 2PXy\partial x & - R\partial P \\ & - 2Qy\partial P & + P\partial R \\ & + 2Qy\partial Q & \end{array}\right\} = 0,$$

vbi ex singulis commode definitur $\frac{\partial P}{P}$: scilicet

$$\frac{\partial P}{P} = 2Q\partial x = \frac{R\partial x - X\partial x + \partial Q}{Q} = \frac{\partial R - 2QX\partial x}{R}.$$

Hinc colligitur $2Q(R+X)\partial x = \partial R$, vnde nunc ipsum elementum ∂x definiamus, $\partial x = \frac{\partial R}{2Q(R+X)}$, quo valore substituto adipiscimur

Q∂R

$$\frac{Q\partial R}{R+X}=\frac{(R-X)\partial R}{2Q(R+X)}+\partial Q \text{ seu}$$

$$2QQ\partial R=R\partial R-X\partial R+2QR\partial Q+2QX\partial Q:$$

vnde colligimus

$$X=\frac{2QQ\partial R-2QR\partial Q-R\partial R}{2Q\partial Q-\partial R}, \text{ et } R+X=\frac{2(QQ-R)\partial R}{2Q\partial Q-\partial R},$$

hinc $\partial x=\frac{2Q\partial Q-\partial R}{4Q(QQ-R)}$, atque $\frac{\partial P}{P}=\frac{2Q\partial Q-\partial R}{2(QQ-R)}$; ideoque

$$P=A\sqrt{(QQ-R)}.$$

Fiat $QQ-R=S$, ac reperietur

$$\partial x=\frac{\partial S}{4QS}, \; X=\frac{2QS\partial Q}{\partial S}-QQ+S, \; R=QQ-S,$$

atque $P=A\sqrt{S}$. Quocirca habebimus hanc aequationem

$$\partial y+\frac{yy\partial S}{4QS}+\partial Q-\frac{(QQ+S)\partial S}{4QS}=0,$$

quae integrabilis redditur per hunc multiplicatorem

$$\frac{\sqrt{S}}{yy+2Qy+QQ-S}=\frac{\sqrt{S}}{(y+Q)^2-S}.$$

Ad eius integrale inueniendum, sumantur Q et S constantes, prodibitque

$$\int\frac{\partial y\sqrt{S}}{(y+Q)^2-S}=\frac{1}{2}l\frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}}+V,$$

existente V certa functione ipsius S vel Q. Iam differentietur haec forma sumta y constante, proditque

$$\frac{\partial Q\sqrt{S}-\frac{(Q+y)\partial S}{2\sqrt{S}}}{(y+Q)^2-S}+\partial V=\frac{yy\partial S+4QS\partial Q-QQ\partial S-S\partial S}{4Q[(y+Q)^2-S]\sqrt{S}},$$

ideoque

$$\partial V=\frac{yy\partial S+2Qy\partial S+QQ\partial S-S\partial S}{4Q[(y+Q)^2-S]\sqrt{S}}=\frac{\partial S}{4Q\sqrt{S}}.$$

Ex quo aequationis nostrae integrale est

$$\frac{1}{2}l\frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}}+\frac{1}{4}\int\frac{\partial S}{Q\sqrt{S}}=C.$$

Corol-

Corollarium 1.

530. Singularis est casus, quo $R = QQ$, fit enim

$$\frac{\partial P}{P} = 2Q\partial x = \frac{QQ\partial x - X\partial x + \partial Q}{Q} = \frac{2\partial Q - 2X\partial x}{Q},$$

vnde has duas aequationes elicimus

$$QQ\partial x + X\partial x - \partial Q = 0 \text{ et } QQ\partial x + X\partial x - \partial Q = 0;$$

quae cum inter se conueniant, erit

$$X\partial x = \partial Q - QQ\partial x, \text{ et } lP = 2\int Q\partial x.$$

Corollarium 2.

531. Sumto ergo Q negatiuo, vt habeamus hanc aequationem

$$\partial y + yy\partial x - \partial Q - QQ\partial x = 0,$$

haec integrabilis redditur, per hunc multiplicatorem

$$\frac{e^{-2\int Q\partial x}}{(y-Q)^2}. \text{ Et integrale erit}$$

$$\frac{-1}{y-Q} e^{-2\int Q\partial x} + V = \text{Const.}$$

vbi V est functio ipsius x, ad quam definiendam, differentietur sumta y constante

$$\frac{-\partial Q}{(y-Q)^2} e^{-2\int Q\partial x} + \frac{2Q\partial x}{y-Q} e^{-2\int Q\partial x} + \partial V = \frac{yy\partial x - \partial Q - QQ\partial x}{(y-Q)^2} e^{-2\int Q\partial x},$$

vnde fit $V = \int e^{-2\int Q\partial x}\partial x$, ita vt integrale sit

$$\int e^{-2\int Q\partial x}\partial x - \frac{e^{-2\int Q\partial x}}{y-Q} = C.$$

Corollarium 3.

532. Proposita ergo aequatione

$$\partial y + yy\partial x + X\partial x = 0,$$

si

ſi eius integrale particulare quoddam conſtet $y = Q$, vt ſit

$$\partial Q + QQ\partial x + X\partial x = 0,$$

ideoque

$$\partial y + yy\partial x - \partial Q - QQ\partial x = 0,$$

multiplicator pro ea erit $\frac{1}{(y-Q)^2} e^{-2\int Q\partial x}$, et integrale completum

$$C e^{2\int Q\partial x} + \frac{1}{(y-Q)} = e^{2\int Q\partial x}\int e^{-2\int Q\partial x}\partial x.$$

Scholion.

533. Aequatio autem in praecedente ſcholio inuenta

$$\partial y + \frac{yy\partial S}{4QS} + \partial Q - \frac{(QQ+S)\partial S}{4QS} = 0,$$

non multum habet in receſſu, poſito enim $y + Q = z$ prodit

$$\partial z - \frac{z\partial S}{2S} + \frac{\partial S(zz-S)}{4QS} = 0,$$

in qua vt bini priores termini in vnum contrahantur, ponatur $z = v\surd S$, reperieturque

$$\partial v\surd S + \frac{vv\partial S}{4Q} - \frac{\partial S}{4Q} = 0, \text{ ſeu } \frac{\partial v}{vv-1} + \frac{\partial S}{4Q\surd S} = 0,$$

quae cum ſit ſeparata integrale erit $\frac{1}{2} l\frac{1+v}{1-v} = \frac{1}{4}\int\frac{\partial S}{Q\surd S}$, vbi eſt $v = \frac{y+Q}{\surd S}$.

Aequatio autem in ipſa ſolutione inuenta

$$\partial y + yy\partial x + \frac{C\partial x}{S^4} - \frac{\partial\partial S}{S\partial x} = 0,$$

vbi S eſt functio quaecunque ipſius x, et $\frac{\partial\partial S}{\partial x} = \partial . \frac{\partial S}{\partial x}$, magis ardua videtur, dum per ſe fit integrabilis, ſi diuidatur per

$$SSyy - \frac{2Sy\partial S}{\partial x} + \frac{\partial S^2}{\partial x^2} + \frac{C}{SS} = (Sy - \frac{\partial S}{\partial x})^2 + \frac{C}{SS}.$$

At ſumto x conſtante integrale reperitur

$$\frac{1}{\surd C}\text{ Arc. tang. }\frac{SSy\partial x - S\partial S}{\partial x\surd C} + V = \text{Conſt.}$$

nunc

nunc ergo ad functionem V inueniendam, sumatur differentiale posita y constante, quod est

$$\frac{2\,Sy\,\partial S - \frac{S\,\partial\partial S}{\partial x} - \frac{\partial S^2}{\partial x}}{SS\,(Sy - \frac{\partial S}{\partial x})^2 + C} + \partial V,$$

et aequari debet alteri parti

$$\frac{\frac{C\,\partial x}{S^4} - \frac{\partial\partial S}{S\,\partial x} + yy\,\partial x}{(Sy - \frac{\partial S}{\partial x})^2 + \frac{C}{SS}} = \frac{\frac{C\,\partial x}{SS} - \frac{S\,\partial\partial S}{\partial x} + SSyy\,\partial x}{SS\,(Sy - \frac{\partial S}{\partial x})^2 + C}.$$

Ergo

$$\partial V = \frac{SSyy\,\partial x - 2\,Sy\,\partial S + \frac{\partial S^2}{\partial x} + \frac{C\,\partial x}{SS}}{SS\,(Sy - \frac{\partial S}{\partial x})^2 + C} = \frac{\partial x}{SS}.$$

Quocirca integrale completum est

$$\frac{1}{\sqrt{C}}\,\text{Arc. tang.}\,\frac{SSy\,\partial x - S\,\partial S}{\partial x\sqrt{C}} + \int\frac{\partial x}{SS} = D.$$

Quod si sumamus $S = x$, huius aequationis

$$\partial y + yy\,\partial x + \frac{C\,\partial x}{x^4} = 0,$$

integrale completum est

$$\frac{1}{\sqrt{C}}\,\text{Arc. tang.}\,\frac{xxy - x}{\sqrt{C}} - \frac{1}{x} = D.$$

Sin autem fit $S = x^n$,

$$\text{ob } \frac{\partial S}{\partial x} = n\,x^{n-1} \text{ et } \partial.\frac{\partial S}{\partial x} = n\,(n-1)\,x^{n-2}\,\partial x,$$

integrari poterit haec aequatio

$$\partial y + yy\,\partial x + \frac{C\,\partial x}{x^{4n}} - \frac{n\,(n-1)\,\partial x}{x.x} = 0,$$

integrale enim erit

$$\frac{1}{\sqrt{C}}\,\text{Arc. tang.}\,\frac{x^{2n}\,y - n\,x^{2n-1}}{\sqrt{C}} - \frac{1}{(2n-1)\,x^{2n-1}} = D.$$

Supra autem inuenimus hanc aequationem

∂y

$$\partial y + yy\,\partial x + C x^m \partial x = 0,$$

ad ſeparationem reduci poſſe, quoties fuerit $m = \frac{-4i}{2i+1}$, iisdem ergo caſibus functionem S aſſignare licebit, vt fiat $\frac{C}{S^4} - \frac{\partial \partial S}{S\,\partial x^2} = C x^m$; quod cum ad aequationes differentiales ſecundi gradus pertineat, hic non attingemus.

Problema 69.

534. Definire functiones P et Q ambarum variabilium x et y, vt aequatio differentialis $P\,\partial x + Q\,\partial y = 0$, diviſa per $Px + Qy$ fiat per ſe integrabilis.

Solutio.

Cum formula $\frac{P\,\partial x + Q\,\partial y}{Px + Qy}$ debeat eſſe integrabilis, ſtatuamus $Q = PR$, vt habeamus $\frac{\partial x + R\,\partial y}{x + Ry}$, ſitque $\partial R = M\,\partial x + N\,\partial y$. Quare fieri oportet

$$\frac{1}{\partial y}\,\partial \cdot \frac{1}{x+Ry} = \frac{1}{\partial x}\,\partial \cdot \frac{R}{x+Ry},$$

vnde nanciſcimur $\frac{-R-Ny}{(x+Ry)^2} = \frac{Mx-R}{(x+Ry)^2}$ ſeu $N = -\frac{Mx}{y}$; hinc fit $\partial R = M\,\partial x - \frac{Mx\,\partial y}{y} = My \cdot \frac{y\,\partial x - x\,\partial y}{yy}$, quae formula cum debeat eſſe integrabilis, neceſſe eſt ſit My functio ipſius $\frac{x}{y}$, quia $\frac{y\,\partial x - x\,\partial y}{yy} = \partial \cdot \frac{x}{y}$; atque ex hac integratione prodit $R = \Phi : \frac{x}{y}$, ſeu quod eodem redit, R erit functio nullius dimenſionis ipſarum x et y. Quocirca cum $\frac{Q}{P} = R$, manifeſtum eſt huic conditioni ſatisfieri, ſi P et Q fuerint functiones homogeneae eiusdem dimenſionum numeri ipſarum x et y; hoc ergo modo eandem integrationem aequationum homogenearum ſumus aſſecuti, quam in capite ſuperiori docuimus.

Corollarium 1.

535. Cum igitur $\frac{\partial t + R\,\partial u}{t + Ru}$ ſit integrabile, ſi fuerit

$R = \Phi : \frac{t}{u}$, seu $R = \frac{t}{u}\Phi : \frac{t}{u}$, erit etiam haec formula $\frac{\frac{\partial t}{t} + \frac{\partial u}{u}\Phi : \frac{t}{u}}{1 + \Phi : \frac{t}{u}}$ integrabilis, quae ita repraesentari potest

$$\frac{\frac{\partial t}{t} + \frac{\partial u}{u}\Phi : (\int\frac{\partial t}{t} - \int\frac{\partial u}{u})}{1 + \Phi : (\int\frac{\partial t}{t} - \int\frac{\partial u}{u})},$$

vbi littera Φ denotat functionem quamcunque quantitatis suffixae.

Corollarium 2.

536. Ponatur $\frac{\partial t}{t} = \frac{\partial x}{X}$ et $\frac{\partial u}{u} = \frac{\partial y}{Y}$, atque haec formula

$$\frac{\frac{\partial x}{X} + \frac{\partial y}{Y}\Phi : (\int\frac{\partial x}{X} - \int\frac{\partial y}{Y})}{1 + \Phi : (\int\frac{\partial x}{X} - \int\frac{\partial y}{Y})} = \frac{\partial x + \frac{X\partial y}{Y}\Phi : (\int\frac{\partial x}{X} - \int\frac{\partial y}{Y})}{X + X\Phi : (\int\frac{\partial x}{X} - \int\frac{\partial y}{Y})}$$

erit per se integrabilis. Quare posito $R = \frac{X}{Y}\Phi : (\int\frac{\partial x}{Y} - \int\frac{\partial y}{Y})$, haec formula $\frac{\partial x + R\partial y}{X + RY}$ erit per se integrabilis, quaecunque functio sit X ipsius x, et Y ipsius y.

Corollarium 3.

537. Quare si quaerantur functiones P et Q, vt haec aequatio $P\partial x + Q\partial y = 0$ fiat integrabilis, si diuidatur per $PX + QY$, existente X functione quacunque ipsius x et Y ipsius y, debet esse $\frac{Q}{P} = \frac{X}{Y}\Phi : (\int\frac{\partial x}{X} - \int\frac{\partial y}{Y})$.

Corollarium 4.

538. Quare si signa Φ et ψ functiones quascunque indicent, fueritque

$$P = \frac{V}{X}\Phi : (\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}) \text{ et } Q = \frac{V}{Y}\psi : (\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}),$$

haec

haec aequatio $P \partial x + Q \partial y = 0$ integrabilis reddetur, si diuidatur per $P X + Q Y$.

Scholion.

539. Hinc ergo innumerabiles aequationes proferri possunt, quas integrare licebit, etiamsi alioquin difficillime pateat, quomodo eae ad separationem variabilium reduci queant. Verum haec inuestigatio proprie ad librum secundum Calculi Integralis est referenda, cuius iam egregia specimina hic habentur; definiuimus enim functionem R binarum variabilium x et y ex certa conditione inter M et N proposita scilicet $M x + N y = 0$ seu $x (\frac{\partial R}{\partial x}) + y (\frac{\partial R}{\partial y}) = 0$, hoc est ex certa differentialium conditione.

CAPVT IV.

DE INTEGRATIONE PARTICVLARI AEQVATIONVM DIFFERENTIALIVM.

Definitio.

540.

Integrale particulare aequationis differentialis eſt relatio variabilium aequationi ſatisfaciens, quae nullam nouam quantitatem conſtantem in ſe complectitur. Opponitur ergo integrali completo, quod conſtantem in differentiali non contentam inuoluit, in quo tamen contineatur neceſſe eſt.

Corollarium 1.

541. Cognito ergo integrali completo, ex eo innumerabilia integralia particularia exhiberi poſſunt, prout conſtanti illi arbitrariae alii atque alii valores determinati tribuuntur.

Corollarium 2.

542. Propoſita ergo aequatione differentiali inter variabiles x et y, omnes functiones ipſius x, quae loco y ſubſtitutae aequationi ſatisfaciunt, dabunt integralia particularia, niſi forte ſint completa.

Corollarium 3.

543. Cum omnis aequatio differentialis ad hanc formam $\frac{\partial y}{\partial x} = V$ reuocetur, exiſtente V functione quacunque ipſarum x et y, ſi eiusmodi conſtet relatio inter x et y, vnde pro

pro $\frac{\partial y}{\partial x}$ et V resultent valores aequales, ea pro integrali particulari erit habenda.

Scholion 1.

544. Interdum facile est integrale particulare quasi diuinatione colligere; veluti si proposita sit haec aequatio

$$aa\partial y + yy\partial x = aa\partial x + xy\partial x.$$

Statim liquet ei satisfieri ponendo $y = x$, quae relatio cum non solum nullam nouam constantem, sed ne eam quidem a, quae in ipsa aequatione differentiali continetur, implicet, vtique est integrale particulare: vnde nihil pro integrali completo colligere licet. Saepe numero quidem cognitio integralis particularis ad inuentionem completi viam patefacit, quemadmodum in hoc ipso exemplo vsu venit, in quo si statuamus $y = x + z$ fit

$$aa\partial x + aa\partial z + xx\partial x + 2xz\partial x + zz\partial x = aa\partial x + xx\partial x + xz\partial x, \text{ seu}$$

$$aa\partial z + xz\partial x + zz\partial x = 0,$$

quae aequatio posito $z = \frac{aa}{v}$ abit in hanc

$$\partial v - \frac{xv\partial x}{aa} = \partial x,$$

quae per $e^{-\int\frac{x\partial x}{aa}} = e^{\frac{-xx}{2aa}}$ multiplicata fit integrabilis, et dat

$$e^{\frac{-xx}{2aa}}v = \int e^{\frac{-xx}{2aa}}\partial x, \text{ seu } v = e^{\frac{xx}{2aa}}\int e^{\frac{-xx}{2aa}}\partial x,$$

quod ergo est maxime transcendens, cum tamen simplicissimum illud particulare inuoluat: scilicet si constans integratione $\int e^{\frac{-xx}{2aa}}\partial x$ inuecta sumatur infinita, fit $v = \infty$ et $z = 0$, vnde $y = x$. Interdum autem integrale particulare parum iuvat ad completum inuestigandum, veluti si habeatur haec aequatio

 $a^3\partial y$

$$a^3 \partial y + y^3 \partial x = a^3 \partial x + x^3 \partial x,$$

cui manifesto satisfacit $y = x$, posito autem $y = x + z$ prodit

$$a^3 \partial z + 3 x x z \partial x + 3 x z z \partial x + z^3 \partial x = 0,$$

cuius resolutio haud facilior videtur, quam illius.

Scholion 2.

545. In his exemplis integrale particulare statim in oculos incurrit, dantur autem casus quibus difficilius perspicitur; et quanquam raro inde via pateat ad integrale completum perueniendi, tamen saepe numero plurimum interest integrale particulare nosse, cum eo nonnunquam totum negotium confici possit. Iam enim animaduertimus in omnibus problematibus, quorum solutio ad aequationem differentialem perducitur, constantem arbitrariam per integrationem inuectam ex ipsis conditionibus, cuique problemati adiunctis, determinari, ita vt semper integrali tantum particulari sit opus; quare si eueniat, vt hoc ipsum integrale particulare cognosci possit, sine subsidio completi, solutio problematis exhiberi poterit, etiamsi integratio aequationis differentialis non sit in potestate. Quibus ergo casibus sine integratione vera solutio inueniri est censenda; propterea quod proprie loquendo nulla aequatio differentialis integrari existimatur, nisi eius integrale completum assignetur. Quocirca vtile erit eos casus perpendere, quibus integrale particulare exhibere licet.

Scholion 3.

546. Maximi autem est momenti hic animaduertisse, non omnes valores aequationi cuipiam differentiali satisfacientes pro eius integrali particulari haberi posse. Veluti si habeatur haec aequatio $\partial y = \frac{\partial x}{\sqrt{(a-x)}}$, seu $\frac{\partial x}{\partial y} = \sqrt{(a-x)}$, posito $x = a$ fit tam $\sqrt{(a-x)} = 0$, quam $\frac{\partial x}{\partial y} = 0$, ita vt aequatio

quatio $x = a$ illi differentiali satisfaciat, cum tamen nequaquam eius sit integrale particulare. Integrale namque completum est $y = C - 2\sqrt{(a - x)}$ seu $a - x = \frac{1}{4}(C - y)^2$, vnde quicunque valor constanti C tribuatur, nunquam sequitur $a - x = 0$. Simili modo huic aequationi

$$\partial y = \frac{x\partial x + y\partial y}{\sqrt{(xx + yy - aa)}}$$

satisfacit haec aequatio finita $xx + yy = aa$, quae tamen inter integralia particularia admitti nequit, propterea quod in integrali completo $y = C + \sqrt{(xx + yy - aa)}$ neutiquam continetur. Quare ad integrale particulare non sufficit, vt eo aequationi differentiali satisfiat, sed insuper hanc conditionem adiungi oportet, vt in integrali completo contineatur; ex quo inuestigatio integralium particularium maxime est lubrica, nisi simul integrale completum innotescat; hoc autem cognito superuacuum esset methodo peculiari in integralia particularia inquirere. Tum enim potissimum iuuat ad inuestigationem integralium particularium confugere, quando integrale completum elicere non licet. Quo igitur hinc fructum percipere queamus, criteria tradi conueniet, ex quibus valores, qui aequationi cuipiam differentiali satisfaciunt, diiudicare liceat, vtrum sint integralia particularia, nec ne? Etiamsi scilicet omnia integralia sint eiusmodi valores, qui aequationi differentiali satisfaciant, tamen non vicissim omnes valores, qui satisfaciunt, sunt integralia. Quod cum parum adhuc sit animaduersum, operam dabo, vt hoc argumentum dilucide euoluam.

Problema 70.

547. Si in aequatione differentiali $\partial y = \frac{\partial x}{Q}$, functio Q euanescat posito $x = a$, determinare quibus casibus haec aequatio $x = a$ sit integrale particulare aequationis differentialis propositae?

Solutio.

Solutio.

Cum ſit $Q = \frac{\partial x}{\partial y}$, poſito $x = a$ fit tam $Q = 0$ quam $\frac{\partial x}{\partial y} = 0$, vnde hic valor $x = a$ aequationi differentiali propoſitae $\partial y = \frac{\partial x}{Q}$ vtique ſatisfacit, neque tamen hinc ſequitur eum eſſe integrale. Hoc ſolum ſcilicet non ſufficit, ſed inſuper requiritur, vt aequatio $x = a$ in integrali completo contineatur, ſi quidem conſtanti per integrationem inuectae certus quidam valor tribuatur. Ponamus ergo P eſſe integrale formulae $\frac{\partial x}{Q}$, vt integrale completum ſit $y = C + P$; cui aequationi ponendo $x = a$ ſatisfieri nequit, niſi poſito $x = a$ fiat $P = \infty$, tum enim ſumta conſtante C pariter infinita, poſitione $x = a$ quantitas y manet indeterminata, ideoque ſi poſito $x = a$ fiat $P = \infty$, tum demum aequatio $x = a$ pro integrali particulari erit habenda. En ergo criterium, ex quo dignoſcere licet, vtrum valor $x = a$ aequationi differentiali $\partial y = \frac{\partial x}{Q}$ ſatisfaciens ſimul ſit eius integrale particuláre nec ne? ſcilicet tum demum erit integrale, ſi poſito $x = a$ non ſolum fiat $Q = 0$, ſed etiam integrale $P = \int \frac{\partial x}{Q}$ abeat in infinitum. Quód quo clarius exponamus, quoniam poſito $x = a$ fit $Q = 0$, ponamus $Q = (a - x)^n R$, denotante n numerum quemcunque poſitiuum, et cum aequatio

$$\partial y = \frac{\partial x}{Q} = \frac{\partial x}{(a-x)^n R}$$

induere queat hanc formam

$$\partial y = \frac{\alpha \partial x}{(a-x)^n} + \frac{\beta \partial x}{(a-x)^{n-1}} + \frac{\gamma \partial x}{(a-x)^{n-2}} + \dots + \frac{S \partial x}{R},$$

ratio illius infiniti P pendebit a termino $\int \frac{\alpha \partial x}{(a-x)^n}$ qui ſi poſito $x = a$ euadat infinitus, etiam integrale $P = \int \frac{\partial x}{Q}$ erit infini-

finitum, vtcunque se habeant reliqua membra. At est

$$\int \frac{a\,\partial x}{(a-x)^n} = \frac{a}{(n-1)\,(a-x)^{n-1}},$$

quae expressio fit infinita posito $x = a$, dummodo $n - 1$ sit numerus positiuus, vel etiam $n = 1$. Quare dummodo exponens n non sit vnitate minor, posito $Q = (a - x)^n R$ aequatio $x = a$ pro integrali particulari erit habenda.

Corollarium 1.

548. Quoties ergo posito $Q = (a - x)^n R$ exponens n est vnitate minor, aequationi $\partial y = \frac{\partial x}{Q}$ non conuenit integrale particulare $x = a$, etiamsi hoc modo aequationi differentiali satisfiat.

Corollarium 2.

549. Si exponens n est vnitate minor, formula $\frac{\partial Q}{\partial x}$ fit infinita posito $x = a$; vnde nouum criterium adipiscimur: Scilicet proposita aequatione $\partial y = \frac{\partial x}{Q}$, si posito $x = a$ fiat quidem $Q = 0$, at $\frac{\partial Q}{\partial x} = \infty$, tum valor $x = a$ non est integrale particulare illius aequationis.

Corollarium 3.

550. His igitur casibus exclusis, aequationis $\partial y = \frac{\partial x}{Q}$, vbi posito $x = a$ fit $Q = 0$, integrale particulare semper erit $x = a$, nisi eodem casu $x = a$ fiat $\frac{\partial Q}{\partial x} = \infty$; hoc est quoties valor formulae $\frac{\partial Q}{\partial x}$ fuerit vel finitus vel euanescat.

Scholion 1.

551. Haec conclusio inuersioni propositionum hypotheticarum innixa licet videri queat suspecta ac regulis Logicae aduersa, verum totum ratiocinium regulis apprime est con-

sentaneum, cum a sublatione consequentis ad sublationem antecedentis concludat. Quoties enim posito $Q = (a - x)^n R$ exponens n est vnitate minor, toties $\frac{\partial Q}{\partial x}$ fit $= \infty$ posito $x = a$. Quare si posito $x = a$ non fiat $\frac{\partial Q}{\partial x} = \infty$, ideoque eius valor vel finitus vel euanescat, tum certe exponens n non est vnitate minor, erit ergo vel maior vnitate vel ipsi aequalis, vtroque autem casu integrale $P = \int \frac{\partial x}{Q}$ posito $x = a$ fit infinitum, ideoque aequatio $x = a$ est integrale particulare. Quare si in aequatione differentiali $\partial y = \frac{\partial x}{Q}$ posito $x = a$ fiat $Q = 0$, examinatur valor $\frac{\partial Q}{\partial x}$ pro casu $x = a$, qui si fuerit vel finitus vel euanescat, aequatio $x = a$ est integrale particulare; sin autem is sit infinitus, ea inter integralia locum non habet, etiamsi aequationi differentiali satisfiat. Eadem regula quoque locum habet, si aequatio differentialis fuerit huiusmodi $\partial y = \frac{P \partial x}{Q}$ seu $\frac{\partial y}{\partial x} = \frac{P}{Q}$, ac posito $x = a$ fiat $Q = 0$, quaecunque fuerit P functio ipsarum x et y; quin etiam necesse non est, vt Q sit functio solius variabilis x, sed simul alteram y vtcunque implicare potest.

Scholion 2.

552. Demonstratio quidem inde est petita, quod quantitas Q, quae posito $x = a$ euanescit, factorem implicet potestatem quampiam ipsius $a - x$, quod in functionibus algebraicis est manifestum. Verum in functionibus transcendentibus eadem regula locum habet, cum potestate talibus dignitatibus aequiualeant. Veluti si sit $\partial y = \frac{\partial x}{lx - la}$, vbi $Q = lx - la = l\frac{x}{a}$, sitque $Q = 0$ posito $x = a$, quaeratur $\frac{\partial Q}{\partial x} = \frac{1}{x}$, quae formula cum non fiat infinita posito $x = a$, integrale particulare erit $x = a$. Quod etiam valet pro aequatione $\partial y = \frac{P \partial x}{lx - la}$, dummodo P non fiat $= 0$ posito $x = a$. Sit enim

enim $P = \frac{1}{x}$, erit integrando $y = C + l(lx - la)$ et $l\frac{x}{a} = e^{y-C}$. Sumta iam constante $C = \infty$, fit $l\frac{x}{a} = o$, ideoque $x = a$, quod ergo est integrale particulare. Simili modo si fit $\partial y = P\partial x : (e^{\frac{x}{a}} - e)$, vbi $Q = e^{\frac{x}{a}} - e$, ideoque posito $x = a$ fit $Q = o$; quia $\frac{\partial Q}{\partial x} = \frac{1}{a}e^{\frac{x}{a}}$, hincque posito $x = a$ fit $\frac{\partial Q}{\partial x} = \frac{e}{a}$, erit $x = a$ etiam integrale particulare. Sumatur $P = e^{\frac{x}{a}}$ vt integratio succedat, et quia $y = C + al(e^{\frac{x}{a}} - e)$, hincque $e^{\frac{x}{a}} = e + e^{\frac{y-C}{a}}$, statuatur $C = \infty$, erit $e^{\frac{x}{a}} = e$, ideoque $x = a$, quod ergo manifesto est integrale particulare.

Exemplum I.

553. *Proposita aequatione differentiali $\partial y = \frac{P\partial x}{\sqrt{S}}$, in qua S euanescat posito $x = a$, definire casus, quibus aequatio $x = a$ est eius integrale particulare.*

Cum hic sit $\sqrt{S} = Q$, erit $\partial Q = \frac{\partial S}{2\sqrt{S}}$: ergo vt integrale particulare sit $x = a$, necesse est, vt posito $x = a$ fiat $\frac{\partial Q}{\partial x} = \frac{\partial S}{2\partial x\sqrt{S}}$ quantitas finita. Hinc eodem casu quantitas $\frac{\partial S^2}{S\partial x^2}$ fieri debet finita, vnde cum S euanescat, etiam $\frac{\partial S^2}{\partial x^2}$ ac proinde $\frac{\partial S}{\partial x}$ euanescere debet: Tum autem posito $x = a$ illius fractionis valor est $\frac{2\partial S\partial\partial S}{\partial S\partial x^2} = \frac{2\partial\partial S}{\partial x^2}$, quem ergo finitum esse oportet, vel $= o$. Quare vt aequatio $x = a$ sit integrale particulare aequationis propositae, hae conditiones requiruntur, primo vt posito $x = a$ fiat $S = o$. Secundum vt fiat $\frac{\partial S}{\partial x} = o$, ac tertio vt huius formulae $\frac{\partial\partial S}{\partial x^2}$ valor prodeat vel finitus, vel $= o$, dummodo ne fiat infinite magnus. Si S sit functio rationalis, haec eo redeunt, vt S factorem habeat $(a - x)^2$ vel potestatem altiorem.

Scholion.

554. Haec resolutio vsum habet in motu corporis ad centrum virium attracti dignoscendo, num in circulo fiat. Si enim distantia corporis a centro ponatur $= x$, et vis centripeta huic distantiae conueniens $= X$, pro tempore t talis reperitur aequatio $\partial t = \frac{x \partial x}{\sqrt{(Exx - c^4 - 2\alpha xx \int X \partial x)}}$, vbi E est constans per praecedentem integrationem ingressa, cuius valor quaeritur, vt hinc aequationi satisfaciat valor $x = a$, quo casu corpus in circulo reuoluetur. Hic ergo est $S = Exx - c^4 - 2\alpha xx \int X \partial x$, vel sumi potest $S = E - \frac{c^4}{xx} - 2\alpha \int X \partial x$. Non solum ergo haec quantitas, sed etiam eius differentiale $\frac{\partial S}{\partial x} = \frac{2c^4}{x^3} - 2\alpha X$ euanescere debet posito $x = a$, neque tamen differentio-differentiale $\frac{\partial \partial S}{\partial x^2} = -\frac{6c^4}{x^4} - \frac{2\alpha \partial X}{\partial x}$ in infinitum abire debet. Inde ergo constans a erit valor ipsius x, ex hac aequatione $\alpha x^3 X = c^4$ resultans, qui est radius circuli, in quo corpus reuolui poterit, dummodo constans E, a qua celeritas pendet, ita fuerit comparata, vt posito $x = a$ fiat $E = \frac{c^4}{aa} + 2\alpha \int X \partial x$; nisi forte eodem casu expressio $\frac{6c^4}{x^4} + \frac{2\alpha \partial X}{\partial x}$ seu saltem haec $\frac{\partial X}{\partial x}$ fiat infinita. Hoc enim si eueniret motus in circulo tolleretur; ad quod ostendendum ponamus $X = b + \sqrt{(a-x)}$, vt $\frac{\partial X}{\partial x} = -\frac{1}{2\sqrt{(a-x)}}$ fiat infinitum posito $x = a$, et aequatio $\alpha x^3 X = c^4$ dabit $\alpha a^3 b = c^4$. Tum vero ob

$$\int X \partial x = bx - \tfrac{2}{3}(a-x)^{\frac{3}{2}} \text{ erit}$$
$$E = \alpha ab + 2\alpha ab = 3\alpha ab,$$

nostraque aequatio fit

$$\partial t = \frac{x \partial x}{\sqrt{[3\alpha abxx - \alpha a^3 b - 2\alpha bx^3 + \frac{4}{3}\alpha xx(a-x)^{\frac{3}{2}}]}}$$

cui valor $x = a$ certe non conuenit tanquam integrale. Fit enim

enim

$$S = a(a-x)[-aab - abx + 2bxx + \tfrac{4}{3}xx\sqrt{(a-x)}]$$

cuius factor cum non fit $(a - x)$ fed tantum $(a - x)^{\frac{3}{2}}$, integrale particulare $x = a$ locum habere nequit.

Exemplum 2.

555. *Propofita aequatione differentiali* $\partial y = \frac{P\,\partial x}{\sqrt[n]{S^m}}$, *in qua S euanefcat pofito* $x = a$, *inuenire cafus quibus integrale particulare eft* $x = a$.

Cum fiat $S = 0$ pofito $x = a$, concipere licet $S = (a - x)^\lambda R$, eritque denominator $\sqrt[n]{S^m} = (a - x)^{\frac{\lambda m}{n}} R^{\frac{m}{n}}$, vnde patet aequationem $x = a$ fore integrale particulare aequationis propofitae, fi fuerit $\frac{\lambda m}{n}$ numerus pofitiuus vnitate maior, feu faltem vnitati aequalis, hoc eft, fi fit vel $\lambda = \frac{n}{m}$ vel $\lambda > \frac{n}{m}$, quae diiudicatio fi S fit functio algebraica, facillime inftituitur. Sin autem fit tranfcendens, vt exponens λ in numeris exhiberi nequeat, vti licebit altera regula: fcilicet, cum fit $\sqrt[n]{S^m} = Q$, erit $\frac{\partial Q}{\partial x} = \frac{m S^{\frac{m-n}{n}} \partial S}{n\,\partial x}$, cuius valor debet effe finitus vel nullus pofito $x = a$, fiquidem integrale fit $x = a$. Sit igitur quoque neceffe eft hoc cafu quantitas $\frac{S^{m-n}\,\partial S^n}{\partial x^n}$ finita. Quaeratur ergo huius formulae valor cafu $x = a$, qui fi prodeat infinite magnus, aequatio $x = a$, non erit integrale, fin autem fit vel finitus vel nullus, erit ea certe integrale particulare aequationis propofitae. Hic duo conftituendi funt cafus, prout fuerit vel $m > n$ vel $m < n$.

I. Si $m > n$, quia posito $x = a$ fit $S^{m-n} = 0$, nisi eodem casu fiat $\frac{\partial S}{\partial x} = \infty$, certe erit $x = a$ integrale. Sin autem fiat $\frac{\partial S}{\partial x} = \infty$, vtrumque euenire potest, vt sit integrale et vt non sit. Ad quod dignoscendum ponatur $\frac{\partial x}{\partial S} = T$, vt nostra formula euadat $\frac{S^{m-n}}{T^n}$, cuius tam numerator, quam denominator euanescit posito $x = a$, ex quo eius valor reducitur ad

$$\frac{(m-n)\,S^{m-n-1}\,\partial S}{n\,T^{n-1}\,\partial T} = \frac{-(m-n)\,S^{m-n-1}\,\partial S^{n+1}}{n\,\partial x^n\,\partial\partial S},$$

qui si sit vel finitus vel nullus, integrale erit $x = a$. Simili modo vlterius progredi licet distinguendo casus $m > n + 1$ et $m < n + 1$.

II. Si $m < n$, formula nostra erit $\frac{\partial S^n}{S^{n-m}\,\partial x^n}$, cuius valor vt fiat finitus, necesse est vt sit $\frac{\partial S}{\partial x} = 0$, ac praeterea, quia numerator ac denominator posito $x = a$ euanescit, formulae nostrae valor erit

$$\frac{n\,\partial S^{n-1}\,\partial\partial S}{(n-m)\,S^{n-m-1}\,\partial S\,\partial x^n} = \frac{n\,\partial S^{n-2}\,\partial\partial S}{(n-m)\,S^{n-m-1}\,\partial x^n},$$

quem finitum esse oportet.

Facillime autem iudicium absoluetur, ponendo statim $x = a + \omega$, cum enim posito $x = a$ fiat $S = 0$, hac substitutione quantitas S semper resolui poterit in huiusmodi formam

$$P\,\omega^\alpha + Q\,\omega^\beta + R\,\omega^\gamma + \text{etc.}$$

cuius tantum vnus terminus $P\,\omega^\alpha$ infimam potestatem ipsius ω complectens spectetur; ac si fuerit vel $\alpha = \frac{n}{m}$ vel $\alpha > \frac{n}{m}$, aequatio $x = a$ certe erit integrale particulare.

Scho-

Scholion.

556. Haec vltima methodus eſt tutiſſima, ac ſemper etiam in formulis tranſcendentibus optimo ſucceſſu adhiberi poteſt. Scilicet propoſita aequatione $\partial y = \frac{P \partial x}{Q}$, in qua poſito $x = a$ fiat $Q = 0$, neque vero etiam numerator P euaneſcat: ſtatuatur $x = a \pm \omega$, et quantitas ω ſpectetur vt infinite parua; vt omnes eius poteſtates prae infima euaneſcant, atque quantitas Q huiusmodi formam $R\omega^\lambda$ accipiet, ex qua patebit niſi exponens λ vnitate fuerit minor, aequationem $x = a$ certe fore integrale particulare aequationis propoſitae. Veluti ſi habeamus $\partial y = \frac{\partial x}{\sqrt{(1 + \operatorname{cof.} \frac{\pi x}{a})}}$, cuius denominator euaneſcit ſumto $x = a$, ob $\operatorname{cof.} \pi = -1$, ponamus $x = a - \omega$, erit

$$\operatorname{cof.} \frac{\pi x}{a} = \operatorname{cof.} (\pi - \frac{\pi \omega}{a}) = -1 + \frac{\pi \pi \omega \omega}{2aa}$$

ob ω infinite paruum, hinc noſtrae aequationis denominator fiet $= \frac{\pi \omega}{a \sqrt{2}}$, vnde concludimus integrale particulare vtique eſſe $x = a$. Non autem foret integrale huius aequationis

$$\partial y = \frac{\partial x}{\sqrt[3]{(1 + \operatorname{cof.} \frac{\pi x}{a})}}.$$

Problema 71.

557. Propoſita aequatione differentiali, in qua variabiles ſunt a ſe inuicem ſeparatae, inueſtigare eius integralia particularia.

Solutio.

Sit propoſita haec aequatio $\frac{\partial x}{X} = \frac{\partial y}{Y}$, in qua X ſit functio ipſius x, et Y ipſius y tantum. Ac primo ponatur $X = 0$, indeque quaerantur valores ipſius x, quorum quisque ſit

fit $x = a$, ita vt posito $x = a$, fiat $X = 0$; tum examinetur valor formulae $\frac{\partial X}{\partial x}$ posito $x = a$, qui nisi fiat infinitus, aequationis propositae integrale particulare certe erit $x = a$. Vel ponatur $x = a + \omega$, spectando ω vt quantitatem infinite parvam, ac si prodeat $X = P\omega^\lambda$, exponens λ, nisi sit vnitate minor, indicabit integrale $x = a$; sin autem sit vnitate minor, aequatio $x = a$ pro integrali non erit habenda.

Simili modo examinetur alterius partis denominator Y, qui si euanescat posito $y = b$, hocque casu formula $\frac{\partial Y}{\partial y}$ non fiat infinita, aequatio $y = b$ erit integrale particulare; quod ergo etiam euenit, si posito $y = b \pm \omega$, prodeat $Y = Q\omega^\lambda$, vbi exponens λ vnitate non sit minor.

Corollarium 1.

558. Nisi ergo membra aequationis separatae fuerint fractiones, quarum denominatores certis casibus euanescant, huiusmodi integralia particularia non dantur; nisi forte in tali aequatione $P\partial x = Q\partial y$, factores P et Q certis casibus fiant infiniti, qui autem casus ad praecedentem facile reducitur.

Corollarium 2.

559. Veluti si habeatur $\partial x \,\mathrm{tang.}\, \frac{\pi x}{2a} = \frac{\partial y}{b - y}$, primo quidem integrale particulare est $y = b$, tum vero quia posito $x = a$ fit $\mathrm{tang.}\, \frac{\pi x}{2a} = \infty$, prius membrum ita exhibeatur $\frac{\partial x}{\mathrm{cot.}\, \frac{\pi x}{2a}}$, cuius denominator posito $x = a - \omega$ fit

$$\mathrm{cot.}\left(\frac{\pi}{2} - \frac{\pi\omega}{2a}\right) = \mathrm{tang.}\, \frac{\pi\omega}{2a} = \frac{\pi\omega}{2a},$$

vbi cum exponens ipsius ω vnitate non sit minor, aequatio $x = a$ erit quoque integrale particulare.

Co-

Corollarium 3.

560. Hinc ergo interdum pro eadem aequatione duo pluraue integralia particularia assignari possunt. Veluti pro hac aequatione $\frac{m\partial x}{a-x} = \frac{n\partial y}{b-y}$ integralia particularia sunt $a - x = 0$ et $b - y = 0$, quae etiam ex integrali completo $(a-x)^m = C(b-y)^n$ consequuntur, illud sumendo $C = 0$, hoc vero sumendo $C = \infty$.

Corollarium 4.

561. Simili modo huius aequationis $\frac{ma\partial x}{aa-xx} = \frac{nb\partial y}{bb-yy}$ quatuor dantur integralia particularia $a + x = 0$, $a - x = 0$, $b + y = 0$, $b - y = 0$. Integrale completum vero est

$$\tfrac{m}{2} l \frac{a+x}{a-x} = \tfrac{1}{2} C + \tfrac{n}{2} l \frac{b+y}{b-y}, \text{ seu}$$
$$\left(\frac{a+x}{a-x}\right)^m = C\left(\frac{b+y}{b-y}\right)^n, \text{ vel}$$
$$(a+x)^m (b-y)^n = C(a-x)^m (b+y)^n,$$

vnde illa sponte fluunt.

Corollarium 5.

562. Hinc patet si fuerit $\partial y = \frac{P\partial x}{(a+x)^\alpha (b+x)^\beta (c+x)^\gamma}$; integralia particularia fore $a + x = 0$, $b + x = 0$, $c + x = 0$, si modo exponentes α, β, γ etc. non fuerint vnitate minores. Quare si Q sit functio rationalis ipsius x, proposita aequatione $\partial y = \frac{P\partial x}{Q}$, omnes factores ipsius Q nihilo aequales positi, praebent integralia particularia.

Scholion 1.

563. Hoc etiam pro factoribus imaginariis valet, etiamsi inde parum lucri nanciscamur. Si enim proposita sit aequatio $\partial y = \frac{a\partial x}{aa+xx}$, ex denominatore $aa + xx$ oriuntur integralia particularia $x = a\sqrt{-1}$ et $x = -a\sqrt{-1}$, quae ex

integrali completo, quod eſt $y = C + \text{Ang. tang.}\ \frac{x}{a}$ minus ſequi videntur. Verum poſito $x = a\sqrt{-1}$ notandum eſt, eſſe Ang. tang. $\sqrt{-1} = \infty\sqrt{-1}$, vnde ſi conſtanti C ſimilis forma ſigno contrario affecta tribuatur, altera quantitas y manet indeterminata, etiamſi ponatur $x = a\sqrt{-1}$, quae poſitio propterea pro integrali particulari eſt habenda. Eſt enim in genere

$$\text{Ang. tang.}\ u\sqrt{-1} = \int\frac{\partial u\sqrt{-1}}{1-uu} = \frac{\sqrt{-1}}{2}\, l\,\frac{1+u}{1-u},$$

vnde poſito $u = +1$ vel $u = -1$, prodit $\infty\sqrt{-1}$, quod infinitum in cauſa eſt, vt integralia aſſignata locum habeant. Quocirca in genere affirmare licet, ſi fuerit $\partial y = \frac{P\partial x}{Q}$, denominatorque Q factorem habeat $(a+x)^\lambda$, cuius exponens λ vnitate non ſit minor, ſemper aequationem $a + x = 0$ fore integrale particulare. Sin autem λ ſit vnitate minor etſi poſitiuus, non erit $a + x = 0$ integrale particulare, etiamſi poſito $x = -a$ aequationi differentiali ſatisfaciat.

Scholion 2.

564. Inſigne hoc eſt paradoxon a nemine adhuc, quantum mihi quidem conſtat, obſeruatum, quod aequationi differentiali eiusmodi valor ſatisfacere queat, qui tamen eius non ſit integrale; atque adeo vix patet, quomodo haec cum ſolita integralium idea conciliari poſſint. Quoties enim propoſita aequatione differentiali eiusmodi relationem variabilium exhibere licet, quae ibi ſubſtituta ſatisfaciat, ſeu aequationem identicam producat, vix cuiquam in mentem venit dubitare, an illa relatio pro integrali ſaltem particulari ſit habenda, cum tamen hinc procliue ſit in errorem delabi. Veluti etiamſi huic aequationi $\partial y\sqrt{(aa - xx - yy)} = x\,\partial x + y\,\partial y$ ſatisfaciat haec aequatio finita $xx + yy = aa$, tamen enormem errorem committeremus, ſi eam pro integrali particulari habere vellemus, propterea quod ea in integrali completo $y = C - \sqrt{(aa - xx - yy)}$ neuti-

neutiquam continetur. Quamobrem etsi omne integrale aequationi differentiali satisfacere debet, tamen non vicissim concludere licet, omnem aequationem finitam, quae satisfaciat, eius esse integrale; verum praeterea requiritur, vt ea certa quadam proprietate sit praedita, cuiusmodi hic exposuimus, et qua demum efficitur, vt in integrali completo contineatur. Hoc autem minime aduersatur verae integralium notioni, quam hic stabiliuimus, neque huiusmodi dubium vnquam in integralia per certas regulas inuenta cadere potest; sed tantum in eiusmodi integralibus, quae diuinando quasi sumus assecuti, locum habet. Saepe numero autem, quando integratio non succedit, diuinationi plurimum tribui solet, tum igitur maxime cauendum est, ne relationem quampiam satisfacientem temere pro integrali particulari proferamus. Quod cum iam in aequationibus separatis simus assecuti, quomodo in omnibus aequationibus differentialibus huiusmodi errores vitari oporteat, sedulo inuestigemus.

Problema 72.

565. Si quaepiam relatio inter binas variabiles satisfaciat aequationi differentiali, definire vtrum ea sit integrale particulare nec ne?

Solutio.

Sit $P\,\partial x = Q\,\partial y$ aequatio differentialis proposita, vbi P et Q sint functiones quaecunque ipsarum x et y, cui satisfaciat relatio quaepiam inter x et y, ex qua fiat $y = X$, functioni scilicet cuidam ipsius x, ita vt si loco y vbique scribatur X, reuera prodeat $P\,\partial x = Q\,\partial y$ seu $\frac{\partial y}{\partial x} = \frac{P}{Q}$. Quaeritur ergo vtrum hic valor $y = X$ pro integrali aequationis propositae haberi possit nec ne? Ad hoc diiudicandum ponatur $y = X + \omega$, fietque $\frac{\partial X}{\partial x} + \frac{\partial \omega}{\partial x} = \frac{P}{Q}$, vbi notetur si esset $\omega = 0$,

fore $\frac{\partial X}{\partial x} = \frac{P}{Q}$. Quare ob ω expressio $\frac{P}{Q}$ hac substitutione reducetur ad $\frac{\partial X}{\partial x}$ vna cum quantitate ita per ω affecta, vt euanescat posito $\omega = 0$. In hoc negotio sufficit ω vt particulam infinite paruam spectasse, cuius ergo potestates altiores prae infima negligere liceat. Ponamus igitur hinc fieri $\frac{P}{Q} = \frac{\partial X}{\partial x} + S\omega^\lambda$, habebiturque $\frac{\partial \omega}{\partial x} = S\omega^\lambda$ seu $\frac{\partial \omega}{\omega^\lambda} = S\,\partial x$. Ex superioribus iam perspicuum est, tum demum fore $y = X$ integrale particulare, seu $\omega = 0$, cum exponens λ fuerit vnitati aequalis vel maior: similis enim hic est ratio ac supra, qua requiritur, vt integrale $\int S\,\partial x = \int \frac{\partial \omega}{\omega^\lambda}$ fiat infinitum casu proposito, quo $\omega = 0$, hoc autem non euenit, nisi λ sit vnitati aequalis, vel > 1. Quodsi ergo aequationi $P\,\partial x = Q\,\partial y$ seu $\frac{\partial y}{\partial x} = \frac{P}{Q}$ satisfaciat valor $y = X$, statuatur $y = X + \omega$, spectata particula ω infinite parua, et inuestigetur hinc forma $\frac{Q}{P} = \frac{\partial X}{\partial x} + S\omega^\lambda$, ex qua nisi sit $\lambda < 1$ concludetur, illum valorem $y = X$ esse integrale particulare aequationis propositae.

Scholion.

566. Cum ω tractetur vt quantitas infinite parua, valor ipsius $\frac{P}{Q}$ posito $y = X + \omega$ per differentiationem commodissime inueniri posse videtur. Cum enim $\frac{P}{Q}$ sit functio ipsarum x et y, statuamus

$$\partial . \frac{P}{Q} = M\,\partial x + N\,\partial y,$$

et quia posito $y = X$, fractio $\frac{P}{Q}$ abit in $\frac{\partial X}{\partial x}$ per hypothesin, si loco y scribatur $X + \omega$, ea in $\frac{\partial X}{\partial x} + N\omega$ transibit, vnde ob exponentem ipsius ω vnitatem sequeretur, aequationem $y = X$ semper esse integrale particulare, quod tamen secus euenire potest.

poteſt. Ex quo patet differentiationem loco ſubſtitutionis adhiberi non poſſe; quod quo clarius oſtendatur, ponamus eſſe $\frac{P}{Q} = \sqrt{(y - X)} + \frac{\partial X}{\partial x}$, vnde poſito $y = X + \omega$ manifeſto oritur $\frac{P}{Q} = \frac{\partial X}{\partial x} + \sqrt{\omega}$. At differentiatione vtentes ponendo

$$\partial . \frac{P}{Q} = M \partial x + N \partial y,$$

fiet $N = \frac{1}{2\sqrt{(y - X)}}$, hincque $\frac{P}{Q} = \frac{\partial X}{\partial x} + N\omega$, quae expreſſio ab illa diſcrepat. Illa ſcilicet aequationem $y = X$ ex integralium numero remouet, haec vero admittere videtur. Verum et hic notandum eſt quantitatem N ipſam poteſtatem ipſius ω negatiue inuoluere; vnde poteſtas ω deprimatur. Quare ne hanc rationem ſpectare opus ſit, ſemper praeſtat vera ſubſtitutione vti, differentiatione ſepoſita. Hoc obſeruato haud difficile erit omnes valores, qui aequationi cuipiam differentiali ſatisfaciunt, diiudicare, vtrum ſint vera integralia nec ne?

Exemplum 1.

567. *Cum huic aequationi*

$$\partial x (1 - y^m)^n = \partial y (1 - x^m)^n,$$

manifeſto ſatisfaciat $y = x$, *vtrum ſit eius integrale particulare nec ne? definire.*

Ponatur $y = x + \omega$, et ſpectato ω vt quantitate minima, eſt $y^m = x^m + m x^{m-1} \omega$, et

$$(1 - y^m)^n = (1 - x^m - m x^{m-1} \omega)^n$$
$$= (1 - x^m)^n - m n x^{m-1} \omega (1 - x^m)^{n-1},$$

vnde aequatio $\frac{\partial y}{\partial x} = \frac{(1 - y^m)^n}{(1 - x^m)^n}$ abit in

$$1 + \frac{\partial \omega}{\partial x} = 1 - \frac{m n x^{m-1} \omega}{1 - x^m},$$

 ſeu

seu $\frac{\partial \omega}{\omega} = -\frac{m n x^{m-1} \partial x}{1 - x^m}$; vbi cum ω habeat dimensionem integram, aequatio $y = x$ certe est integrale particulare aequationis differentialis propositae.

Exemplum 2.

568. *Cum huic aequationi*

$$a \partial y - a \partial x = \partial x \sqrt{(yy - xx)},$$

satisfaciat valor $y = x$, inuestigare, vtrum is sit eius integrale particulare nec ne?

Ponatur $y = x + \omega$, et sumta ω quantitate infinite parva, cum sit $\sqrt{(yy - xx)} = \sqrt{2 x \omega}$, erit $a \partial \omega = \partial x \sqrt{2 x \omega}$ seu $\frac{a \partial \omega}{\sqrt{\omega}} = \partial x \sqrt{2 x}$. Quoniam igitur hic $\partial \omega$ diuiditur per potestatem ipsius ω, cuius exponens est vnitate minor, sequitur valorem $y = x$ non esse integrale particulare aequationis propositae, etiamsi ei satisfaciat. Scilicet si eius integrale completum exhibere liceret, pateret, quomodocunque constans arbitraria per integrationem ingressa definiretur, in ea aequationem $y = x$ non contentum iri.

Scholion.

569. Hinc noua ratio intelligitur, cur diiudicatio integralis ab exponente ipsius ω pendeat. Cum enim in exemplo proposito facto $y = x + \omega$ prodeat $\frac{a \partial \omega}{\sqrt{\omega}} = \partial x \sqrt{2 x}$, erit integrando $2 a \sqrt{\omega} = C + \frac{2}{3} x \sqrt{2 x}$. Verum per hypothesin ω est quantitas infinite parua, hinc autem vtcunque definiatur constans C, quantitas ω obtinet valorem finitum, qui adeo quantumuis magnus euadere potest, quod cum hypothesi aduersetur, necessario sequitur aequationem $y = x$ integrale esse non posse; hocque semper euenire debere, quoties $\partial \omega$ prodit diuisum

diuiſum per poteſtatem ipſius ω, cuius exponens vnitate eſt minor. Contra vero patet, ſi facta ſubſtitutione expoſita prodeat $\frac{\partial \omega}{\omega} = R \partial x$, vt poſito $\int R \partial x = l S$ fiat $l \omega = l C + l S$, ſeu $\omega = C S$, ſumta conſtante C euaneſcente vtique ipſam quantitatem ω euaneſcere; quod idem euenit ſi prodeat $\frac{\partial \omega}{\omega^{\lambda}} = R \partial x$, exiſtente $\lambda > 1$. Erit enim $\frac{1}{(\lambda - 1)\,\omega^{\lambda - 1}} = C - S$ ſeu $(\lambda - 1)\,\omega^{\lambda - 1} = \frac{1}{C - S}$, vnde ſumto $C = \infty$, quantitas ω reuera fit euaneſcens, vt hypotheſis exigit.

Caeterum aequatio huius exempli, poſito $x = pp - qq$ et $y = pp + qq$, ab irrationalitate liberatur, fitque $4 a q \partial q = 4 p q (p \partial p - q \partial q)$, ſiue $a \partial q = pp \partial p - p q \partial q$, quae nullo modo tractari poſſe videtur; neque ergo eius integrale completum exhiberi poteſt. Cui aequationi cum non amplius ſatisfacit $x = y$ ſeu $q = 0$, hinc quoque concludendum eſt, valorem $y = x$ non eſſe integrale particulare.

Exemplum 3.

570. *Cum huic aequationi*

$$a a \partial y - a a \partial x = \partial x (yy - xx),$$

ſatisfaciat valor $y = x$, inueſtigare, vtrum is ſit eius integrale particulare nec ne?

Ponatur $y = x + \omega$ ſpectata ω vt quantitate infinite parua, et ob $yy - xx = 2 x \omega$ aequatio noſtra hanc induet formam $a a \partial \omega = 2 x \omega \partial x$, ſeu $\frac{a a \partial \omega}{\omega} = 2 x \partial x$. Quia igitur hic $\partial \omega$ diuiditur per poteſtatem primam ipſius ω, aequatio $y = x$ vtique erit integrale particulare aequationis propoſitae, atque adeo etiam in integrali completo continetur. Hoc enim inuenitur ponendo $y = x - \frac{a a}{u}$, quo fit

$a^4 \partial u$

$$\frac{a^4 \partial u}{uu} = \partial x \left(\frac{a^4}{uu} - \frac{2aax}{u}\right), \text{ seu } \partial u + \frac{2ux\partial x}{aa} = \partial x.$$

Multiplicetur per $e^{\frac{xx}{aa}}$, et integrale prodit

$$e^{\frac{xx}{aa}} u = C + \int e^{\frac{xx}{aa}} \partial x, \text{ hincque}$$

$$y = x - aa e^{\frac{xx}{aa}} : (C + \int e^{\frac{xx}{aa}} \partial x).$$

Quodſi ergo conſtans C capiatur infinita, fit $y = x$.

Scholion.

571. Si in hac aequatione vt ſupra ponatur $x = pp - qq$ et $y = pp + qq$, oritur $aa\partial q = ppq(p\partial p - q\partial q)$, cui ſatisfacit $q = 0$, vnde caſus $y = x$ naſcitur. At facta hac transformatione difficulter patet, quomodo eius integrale inueniri oporteat. Si quidem ſuperiorem reductionem perpendamus, intelligemus hanc aequationem integrabilem reddi ſi multiplicetur per $e^{(pp-qq)^2:aa}:q^3$, quod cum per ſe haud facile pateat, conſultum erit hac ſubſtitutione vti $pp - qq = rr$, qua fit $pp = qq + rr$ et $p\partial p - q\partial q = r\partial r$, vnde aequatio abit in $aa\partial q = qr\partial r(qq + rr)$, ſeu $\frac{aa\partial q}{q^3} = r\partial r + \frac{r^3\partial r}{qq}$, quae poſito $\frac{1}{qq} = s$ facile integratur. Quoties ergo licet eiusmodi relationem inter variabiles colligere, quae aequationi differentiali ſatisfaciat, hoc modo iudicari poterit, vtrum ea relatio pro integrali particulari ſit habenda nec ne? Pro inventione autem huiusmodi integralium particularium regulae vix tradi poſſunt; quae enim habentur regulae, aeque ad integralia completa inuenienda patent. Ita quae ſupra circa aequationes ſeparatas obſeruauimus, ob id ipſum quod ſunt ſeparatae, via ſimul ad integrale completum eſt patefacta. Simili modo ſi altera methodus per factores ſuccedat, plerumque ex ipſis factoribus, quibus aequatio integrabilis redditur, integralia

gralia particularia concludi possunt; quaemadmodum in sequentibus propositionibus declarabimus.

Theorema.

572. Si aequatio differentialis $P\partial x + Q\partial y = 0$ per functionem M multiplicata reddatur integrabilis, integrale particulare erit $M = 0$, nisi eodem casu P vel Q abeat in infinitum.

Demonstratio.

Ponamus u esse factorem ipsius M, et ostendendum est aequationem $u = 0$ esse integrale particulare aequationis propositae. Cum u aequetur certae functioni ipsarum x et y, definiatur inde altera variabilis y, vt aequatio prodeat inter binas variabiles x et u, quae sit $R\partial x + S\partial u = 0$, vnde posito multiplicatore $M = Nu$, integrabilis erit haec forma

$$NRu\partial x + NSu\partial u = 0.$$

Quodsi iam neque R neque S per u diuidatur, quo casu posito $u = 0$ neque P neque Q abit in infinitum, integrale vtique per u erit diuisibile. Nam siue id colligatur ex termino $NRu\partial x$ spectata u vt constante, siue ex termino $NSu\partial u$ spectata x constante, integrale prodit factorem u implicans, si quidem in integratione constans omittatur. Vnde concludimus integrale completum huiusmodi formam esse habiturum $V = uC$. Quare si haec constans C nihilo aequalis capiatur, integrale particulare erit $u = 0$, iis scilicet casibus exceptis, quibus functiones R et S iam ipsae per u essent diuisae, ideoque ratiocinium nostrum vim suam amitteret. His ergo casibus exclusis, quoties aequatio $P\partial x + Q\partial y = 0$ per functionem M multiplicata fit per se integrabilis, eaque functio M factorem habeat u, integrale particulare erit $u = 0$, quod similiter de singulis factoribus functionis M valet.

Scholion.

573. Limitatio adiecta absolute est necessaria, cum ea neglecta vniuersum ratiocinium claudicet. Quod quo facilius intelligatur, consideremus hanc aequationem

$$\frac{a\,\partial x}{y-x} + \partial y - \partial x = 0,$$

quae per $y - x$ multiplicata manifesto fit integrabilis: ponamus ergo hunc multiplicatorem $y - x = u$, seu $y = x + u$, vnde nostra aequatio erit $\frac{a\,\partial x}{u} + \partial u = 0$, quae per u multiplicata, abit in $a\,\partial x + u\,\partial u = 0$: vbi cum pars $a\,\partial x$ non per u sit multiplicata, neutiquam concludere licet integrale per u fore diuisibile, quippe quod est $a x + \frac{1}{2} u u$. Hinc patet, si modo pars ∂x per u esset multiplicata, etiamsi altera pars ∂u factore u careret, tamen integrale per u diuisibile fore, veluti euenit in $u\,\partial x + x\,\partial u$, cuius integrale $x u$ vtique factorem habet u. Ex quo intelligitur, si formula $P u \partial x + Q \partial u$ fuerit per se integrabilis, dummodo Q non diuidatur per u vel per potestatem eius prima altiorem, etiam integrale, omissa scilicet constante fore per u diuisibile.

Theorema.

574. Si aequatio differentialis $P\,\partial x + Q\,\partial y = 0$ per functionem M diuisa euadat per se integrabilis, integrale particulare erit $M = 0$, nisi posito $M = 0$ vel P vel Q euanescat.

Demonstratio.

Habeat diuisor M factorem u, vt sit $M = N u$, et ostendi oportet, integrale particulare futurum $u = 0$, id quod de singulis factoribus diuisoris M, si quidem plures habeat, est tenendum. Cum igitur u sit functio ipsarum x et y, definiatur inde altera y per x et u, vt prodeat huiusmodi aequatio

$R\,\partial x$

$R\partial x + S\partial u = 0$, quae ergo per Nu diuisa per se erit integrabilis. Quaeri igitur oportet integrale formulae $\frac{R\partial x}{Nu} + \frac{S\partial u}{Nu}$, vbi assumimus neque R neque S per u multiplicari, neque hoc modo factorem u ex denominatore tolli. Quod si iam hoc integrale ex solo membro $\frac{R\partial x}{Nu}$ colligatur, spectando u vt constantem, prodit id $\frac{1}{u}\int\frac{R\partial x}{N} + \Phi : u$; sin autem ex altero membro $\frac{S\partial u}{Nu}$ sumta x constante colligatur, quia S non factorem habet u, id semper ita erit comparatum, vt posito $u = 0$, fiat infinitum. Ex quo integrale, quod sit V, ita erit comparatum, vt fiat $= \infty$ posito $u = 0$; quare cum integrale completum futurum sit $V = C$, huic aequationi, sumta constante C infintia, satisfit ponendo $u = 0$. Concludimus itaque, si diuisor $M = Nu$ reddat aequationem differentialem $P\partial x + Q\partial y = 0$ per se integrabilem, ex quolibet diuisoris M factore u obtineri integrale particulare $u = 0$, nisi forte posito $u = 0$, quantitates P et Q, vel R et S euanescant.

Corollarium 1.

575. Si aequatio $P\partial x + Q\partial y = 0$ fuerit homogenea, ea vt supra vidimus integrabilis redditur, si diuidatur per $Px + Qy$, quare integrale eius particulare erit $Px + Qy = 0$. Quae aequatio cum etiam sit homogenea, factores habebit formae $\alpha x + \beta y$, quorum quisque nihilo aequatus dabit integrale particulare.

Corollarium 2.

576. Pro hac aequatione

$$y\partial x(c + nx) - \partial y(y + a + bx + nxx) = 0$$

diuisorem, quo integrabilis redditur, supra §. 488. exhibuimus, vnde integrale particulare concluditur $y = 0$, tum vero

$$nyy+(2na-bc)y+n(b-2c)xy$$
$$+(na+cc-bc)(a+bx+nxx)=0,$$

cuius radices sunt

$$ny=\tfrac{1}{2}bc-na+n(c-\tfrac{1}{2}b)x\pm(c+nx)\sqrt{(\tfrac{1}{4}bb-na)}.$$

Corollarium 3.

577. Pro hac aequatione differentiali

$$\frac{n\partial x(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}+(x-y)\partial y=0$$

diuisorem, quo integrabilis redditur, supra §. 489. dedimus, vnde integrale particulare concludimus

$$x-y+n\sqrt{(1+xx)(1+yy)}=0,\ \text{seu}$$
$$yy-2xy+xx=nn+nnxx+nnyy+nnxxyy,$$

ex quo porro fit $y=\frac{x\pm n(1+xx)\sqrt{(1-nn)}}{1-nn(1+xx)}$.

Corollarium 4.

578. Pro hac aequatione differentiali

$$\partial y+yy\partial x-\frac{a\partial x}{x^4}=0$$

multiplicatorem supra §. 491. inuenimus $\frac{xx}{xx(1-xy)^2-a}$, vnde integrale particulare concludimus $xx(1-xy)^2-a=0$, hincque $x(1-xy)=\pm\sqrt{a}$, seu $y=\frac{1}{x}\pm\frac{\sqrt{a}}{xx}$, ita vt bina habeamus integralia particularia, quae autem imaginaria euadunt, si a fuerit quantitas negatiua.

Scholion.

579. Haec fere sunt omnia, quae circa tractationem aequationum differentialium adhuc sunt explorata, nonnulla tamen subsidia euolutio aequationum differentialium secundi gradus infra suppeditabit. Huc autem commode referri possunt, quae circa comparationem certarum formularum transcendentium haud ita pridem sunt inuestigata. Quemadmodum enim logarithmi

rithmi et arcus circulares, etsi sunt quantitates transcendentes, inter se comparari atque adeo aeque ac quantitates algebraicae in calculo tractari possunt, ita similem comparationem inter certas quantitates transcendentes altioris generis instituere licet, quae scilicet continentur in formula hac

$$\int \frac{\partial x}{\sqrt{(A + Bx + Cx^2 + Dx^3 + Ex^4)}},$$

vbi etiam numerator rationalis veluti $\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 +$ etc. addi potest. Quod argumentum cum sit maxime arduum, atque adeo vires Analyseos superare videatur, nisi certa ratione expediatur, in Analysin inde haud spernenda incrementa redundant; imprimis autem resolutio aequationum differentialium non mediocriter perfici videtur. Cum enim proposita fuerit huiusmodi aequatio

$$\frac{\partial x}{\sqrt{(A + Bx + Cx^2 + Dx^3 + Ex^4)}} = \frac{\partial y}{\sqrt{(A + By + Cy^2 + Dy^3 + Ey^4)}},$$

statim quidem patet eius integrale particulare $x = y$, verum integrale completum maxime transcendens fore videtur, cum vtraque formula per se neque ad logarithmos, neque ad arcus circulares reduci queat. Quare eo magis erit mirandum, quod integrale completum per aequationem adeo algebraicam inter x et y exhiberi possit. Quo autem methodus ad haec sublimia ducens clarius perspiciatur, eam primo ad quantitates transcendentes notas, hac formula $\int \frac{\partial y}{\sqrt{(A + Bx + Cxx)}}$ contentas applicemus, deinceps eius vsum in formulis illis magis complexis ostensuri.

CAPVT V.

DE COMPARATIONE QVANTITATVM TRANSCENDENTIVM IN FORMA $\int \frac{P\partial x}{\sqrt{(A+2Bx+Cxx)}}$ CONTENTARVM.

Problema 73.

580.

Propofita inter x et y hac aequatione algebraica

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy = 0$$

inuenire formulas integrales formae praefcriptae, quae inter fe comparari queant.

Solutio.

Differentietur aequatio propofita, et ex eius differentiali

$$2\beta\partial x + 2\beta\partial y + 2\gamma x\partial x + 2\gamma y\partial y + 2\delta x\partial y + 2\delta y\partial x = 0$$

colligetur haec aequatio

$$\partial x(\beta + \gamma x + \delta y) + \partial y(\beta + \gamma y + \delta x) = 0.$$

Statuatur $\beta + \gamma x + \delta y = p$ et $\beta + \gamma y + \delta x = q$, atque ex priori erit

$$pp = \beta\beta + 2\beta\gamma x + 2\beta\delta y + \gamma\gamma xx + 2\gamma\delta xy + \delta\delta yy,$$

a qua fubtrahatur aequatio propofita per γ multiplicata

$$0 = \alpha\gamma + 2\beta\gamma x + 2\gamma\beta y + \gamma\gamma xx + \gamma\gamma yy + 2\gamma\delta xy,$$

fietque

$$pp = \beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)y + (\delta\delta - \gamma\gamma)yy.$$

Simi-

Similique modo reperietur

$$qq = \beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)x + (\delta\delta - \gamma\gamma)xx,$$

vnde erit $p\,\partial x + q\,\partial y = 0$. Cum iam sit p functio ipsius y, et q similis functio ipsius x, ponatur

$$\beta\beta - \alpha\gamma = A,\ \beta(\delta - \gamma) = B, \text{ et } \delta\delta - \gamma\gamma = C;$$

vnde colligitur

$$\delta - \gamma = \frac{B}{\beta} \text{ et } \delta + \gamma = \frac{C}{\delta - \gamma} = \frac{\beta C}{B},$$

hincque

$$\delta = \frac{BB + \beta\beta C}{2B\beta} \text{ et } \gamma = \frac{\beta\beta C - BB}{2B\beta}:$$

prima vero dat

$$\alpha = \frac{\beta\beta - A}{\gamma} = \frac{2B\beta(\beta\beta - A)}{\beta\beta C - BB}.$$

Quibus valoribus pro α, γ, δ assumtis, aequatio $\frac{\partial x}{q} + \frac{\partial y}{p} = 0$ abit in hanc

$$\frac{\partial x}{\sqrt{(A + 2Bx + Cxx)}} + \frac{\partial y}{\sqrt{(A + 2By + Cyy)}} = 0;$$

cui ergo aequationi differentiali satisfacit aequatio

$$\frac{2B\beta(\beta\beta - A)}{\beta\beta C - BB} + 2\beta(x + y) + \frac{\beta\beta C - BB}{2B\beta}(xx + yy)$$
$$+ \frac{BB + \beta\beta C}{B\beta}xy = 0,$$

quae cum contineat constantem nouam β, erit adeo integrale completum aequationis differentialis inuentae.

Neque vero opus est, vt formulae illae ipsis litteris A, B, C aequentur, sed sufficit vt ipsis sint proportionales, vnde fit

$$\frac{\beta\beta - \alpha\gamma}{\beta(\delta - \gamma)} = \frac{A}{B} \text{ et } \frac{\delta + \gamma}{\beta} = \frac{C}{B}.$$

Ergo

$$\delta = \frac{\beta C}{B} - \gamma \text{ et } \alpha = \frac{\beta\beta}{\gamma} - \frac{\beta A}{\gamma B}(\delta - \gamma), \text{ seu}$$
$$\alpha = \frac{\beta\beta}{\gamma} - \frac{\beta\beta AC}{\gamma BB} + \frac{2\beta A}{B}.$$

Quare

Quare aequationis differentialis

$$\frac{\partial x}{\sqrt{(A+2Bx+Cxx)}}+\frac{\partial y}{\sqrt{(A+2By+Cyy)}}=0$$

integrale completum eſt

$$\beta\beta(CB-AC)+2\beta\gamma AB+2\beta\gamma BB(x+y)+\gamma\gamma BB(xx+yy) + 2\gamma B(\beta C-\gamma B)xy=0,$$

vbi ratio $\frac{\beta}{\gamma}$ conſtantem arbitrariam exhibet.

Corollarium 1.

581. Ex aequatione propoſita radicem extrahendo fit

$$y=\frac{-\beta-\delta x+\sqrt{(\beta\beta+2\beta\delta x+\delta\delta xx-\alpha\gamma-2\beta\gamma x-\gamma\gamma xx)}}{\gamma},$$

ſeu loco α et δ ſubſtitutis valoribus,

$$y=-\frac{\beta}{\gamma}-\frac{(\beta C-\gamma B)}{\gamma B}x+\sqrt{\left(\frac{\beta\beta C-2\beta\gamma B}{\gamma\gamma BB}\right)}(A+2Bx+Cxx).$$

Corollarium 2.

582. Si ergo $x=0$, fit

$$y=-\frac{\beta}{\gamma}+\sqrt{\frac{\beta\beta AC-2\beta\gamma AB}{\gamma\gamma BB}},$$

ponatur hic valor $=a$, vt fit

$$\gamma Ba+\beta B=\sqrt{(\beta\beta AC-2\beta\gamma AB)},$$

vnde ſumtis quadratis oritur

$$\gamma\gamma BBaa+2\beta\gamma BBa+\beta\beta BB=\beta\beta AC-2\beta\gamma AB,$$

hincque

$$\frac{\gamma}{\beta}=\frac{-A-Ba+\sqrt{A(A+2Ba+Caa)}}{Baa},\ \text{ſeu}$$

$$\frac{\beta}{\gamma}=\frac{B(A+Ba+\sqrt{A(A+2Ba+Caa)}}{AC-BB}.$$

Scholion 1.

283. Vt aequatio aſſumta

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0$$

ſatis-

fatisfaciat aequationi differentiali

$$\frac{\partial x}{\sqrt{(A + 2Bx + Cxx)}} + \frac{\partial y}{\sqrt{(A + 2By + Cyy)}} = 0,$$

necesse est vt sit

$$\beta\beta - \alpha\gamma = m\,A,\ \beta(\delta - \gamma) = m\,B \text{ et } \delta\delta - \gamma\gamma = m\,C,$$

vnde fit

$$\beta + \gamma y + \delta x = \sqrt{m}\,(A + 2Bx + Cxx) \text{ et}$$
$$\beta + \gamma x + \delta y = \sqrt{m}\,(A + 2By + Cyy).$$

At ex datis A, B, C, litterarum α, β, γ, δ et m tres tantum definiuntur; quare cum binae maneant indeterminatae, aequatio assumta, etiamsi per quemuis coefficientium diuidatur, vnam tamen constantem continet nouam, ex quo ea pro integrali completo erit habenda. Quare etsi aequationis differentialis neutra pars integrationem algebraice admittit, tamen integrale completum algebraice exhiberi potest. Loco constantis arbitrariae is valor ipsius y introduci potest, quem recipit posito $x = 0$: cum autem euenire possit, vt hic valor fiat imaginarius, conueniet istam constantem ita definiri, vt posito $x = a$ fiat $y = b$, quo pacto ad omnes casus applicatio fieri poterit. Hinc erit

$$\frac{\beta + \gamma b + \delta a}{\beta + \gamma a + \delta b} = \sqrt{\frac{A + 2Ba + Caa}{A + 2Bb + Cbb}},$$

vnde colligitur

$$\beta = \frac{(\gamma a + \delta b)\sqrt{(A + 2Ba + Caa)} - (\gamma b + \delta a)\sqrt{(A + 2Bb + Cbb)}}{-\sqrt{(A + 2Ba + Caa)} + \sqrt{(A + 2Bb + Cbb)}} \text{ et}$$

$$\sqrt{m}\,(A + 2Ba + Caa) = \frac{(\delta - \gamma)(b - a)\sqrt{(A + 2Ba + Caa)}}{\sqrt{(A + 2Bb + Cbb)} - \sqrt{(A + 2Ba + Caa)}}$$

seu

$$\sqrt{m} = \frac{(\delta - \gamma)(b - a)}{\sqrt{(A + 2Bb + Cbb)} - \sqrt{(A + 2Ba + Caa)}}.$$

Ponatur breuitatis gratia

$$\sqrt{(A + 2Ba + Caa)} = \mathfrak{A} \text{ et } \sqrt{(A + 2Bb + Cbb)} = \mathfrak{B},$$

vt sit

$$\sqrt{m} = \frac{(\delta - \gamma)(b - a)}{\mathfrak{B} - \mathfrak{A}} \text{ et}$$

$$\beta = \frac{\mathfrak{A}(\gamma a + \delta b) - \mathfrak{B}(\gamma b + \delta a)}{\mathfrak{B} - \mathfrak{A}},$$

et aequatio $\beta(\delta - \gamma) = m\mathrm{B}$ induet hanc formam

$$\mathfrak{A}(\gamma a + \delta b) - \mathfrak{B}(\gamma b + \delta a) = \frac{\mathrm{B}(\delta - \gamma)(b - a)^2}{\mathfrak{B} - \mathfrak{A}}$$

vnde fit

$$\left.\begin{array}{l} +\gamma\mathfrak{A}\mathfrak{B} - \gamma\mathrm{A} - \gamma\mathrm{B}(a+b) - \gamma\mathrm{C}(aa - ab + bb) \\ +\delta\mathfrak{A}\mathfrak{B} - \delta\mathrm{A} - \delta\mathrm{B}(a+b) - \delta\mathrm{C}ab \end{array}\right\} = 0.$$

Statuatur ergo

$$\gamma = n\mathfrak{A}\mathfrak{B} - n\mathrm{A} - n\mathrm{B}(a+b) - n\mathrm{C}ab$$

$$\delta = n\mathrm{A} + n\mathrm{B}(a+b) + n\mathrm{C}(aa - ab + bb) - n\mathfrak{A}\mathfrak{B}$$

$$\sqrt{m} = \frac{n(b-a)(\mathfrak{A}^2 + \mathfrak{B}^2 - 2\mathfrak{A}\mathfrak{B})}{\mathfrak{B} - \mathfrak{A}} = n(b-a)(\mathfrak{B} - \mathfrak{A})$$

$$\beta = n\mathrm{B}(b-a)^2, \text{ ergo } \delta - \gamma = \frac{m}{n(b-a)^2},$$

vnde cum fit $\delta + \gamma = n\mathrm{C}(b-a)^2$, erit vtique $\delta\delta - \gamma\gamma = m\mathrm{C}$. Superest vt fiat $\alpha\gamma = \beta\beta - m\mathrm{A}$, hoc est

$$\alpha\gamma = nn\mathrm{BB}(b-a)^4 - nn\mathrm{A}(b-a)^2(\mathfrak{B} - \mathfrak{A})^2 \text{ seu}$$

$$\alpha\gamma = nn(b-a)^2[\mathrm{BB}(b-a)^2 - \mathrm{A}(\mathfrak{B} - \mathfrak{A})^2].$$

Vel cum posito $x = a$ fiat $y = b$, erit quoque

$$\alpha = -2\beta(a+b) - \gamma(aa + bb) - 2\delta ab,$$

hincque

$$\alpha = n(a-b)^2[\mathrm{A} - \mathrm{B}(a+b) - \mathrm{C}ab - \mathfrak{A}\mathfrak{B}];$$

vnde aequatio nostra assumta est

$$\begin{array}{l} (b-a)^2[\mathrm{A} - \mathrm{B}(a+b) - \mathrm{C}ab - \mathfrak{A}\mathfrak{B}] + 2\mathrm{B}(b-a)^2(x+y) \\ \quad - [\mathrm{A} + \mathrm{B}(a+b) + \mathrm{C}ab - \mathfrak{A}\mathfrak{B}](xx + yy) \\ \quad + 2[\mathrm{A} + \mathrm{B}(a+b) + \mathrm{C}(aa - ab + bb) - \mathfrak{A}\mathfrak{B}]xy = 0. \end{array}$$

Scholion 2.

584. Si ponatur $\beta = 0$, vt aequatio fit

$\alpha +$

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0, \text{ erit}$$
$$y = \frac{-\delta x + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma)xx]}}{\gamma}.$$

Posito ergo $-\alpha\gamma = m\mathrm{A}$ et $\delta\delta - \gamma\gamma = m\mathrm{C}$, vt sit

$$\gamma y + \delta x = \sqrt{m(\mathrm{A} + \mathrm{C}xx)}, \text{ erit}$$
$$\frac{\partial x}{\sqrt{(\mathrm{A} + \mathrm{C}xx)}} + \frac{\partial y}{\sqrt{(\mathrm{A} + \mathrm{C}yy)}} = 0,$$

cuius aequationis integrale completum erit ipsa aequatio assumta, pro qua habebitur $\frac{\mathrm{C}}{\mathrm{A}} = \frac{\gamma\gamma - \delta\delta}{\alpha\gamma}$, seu $\delta = \sqrt{(\gamma\gamma - \frac{\alpha\gamma\mathrm{C}}{\mathrm{A}})}$. Sin autem posito $x = 0$ fieri debeat $y = b$, ob $\gamma b = \sqrt{m\mathrm{A}}$, erit $\gamma = \frac{\sqrt{m\mathrm{A}}}{b}$; tum $\alpha = -b\sqrt{m\mathrm{A}}$ et $\delta = \sqrt{(\frac{m}{bb} + m\mathrm{C})}$. Habebitur ergo haec aequatio

$$\frac{y\sqrt{m\mathrm{A}}}{b} + \frac{x\sqrt{m(\mathrm{A} + \mathrm{C}bb)}}{\mathrm{A}} = \sqrt{m(\mathrm{A} + \mathrm{C}xx)},$$

quae praebet

$$y = -x\sqrt{\frac{\mathrm{A} + \mathrm{C}bb}{\mathrm{A}}} + b\sqrt{\frac{\mathrm{A} + \mathrm{C}xx}{\mathrm{A}}},$$

quae est integrale completum aequationis illius differentialis. Quare si x capiatur negatiue, huius aequationis differentialis

$$\frac{\partial x}{\sqrt{(\mathrm{A} + \mathrm{C}xx)}} = \frac{\partial y}{\sqrt{(\mathrm{A} + \mathrm{C}yy)}},$$

integrale completum est

$$y = x\sqrt{\frac{\mathrm{A} + \mathrm{C}bb}{\mathrm{A}}} + b\sqrt{\frac{\mathrm{A} + \mathrm{C}xx}{\mathrm{A}}}.$$

Quodsi simili modo calculus in genere tractetur, aequationis differentialis

$$\frac{\partial x}{\sqrt{(\mathrm{A} + 2\mathrm{B}x + \mathrm{C}xx)}} + \frac{\partial y}{\sqrt{(\mathrm{A} + 2\mathrm{B}y + \mathrm{C}yy)}} = 0,$$

si breuitatis gratia ponatur $\sqrt{(\mathrm{A} + 2\mathrm{B}b + \mathrm{C}bb)} = \mathfrak{B}$, erit integrale completum

$$y\left(\sqrt{\mathrm{A}} + \frac{\mathrm{B}b}{\sqrt{\mathrm{A}} - \mathfrak{B}}\right) + x\left(\mathfrak{B} + \frac{\mathrm{B}b}{\sqrt{\mathrm{A}} - \mathfrak{B}}\right)$$
$$= \frac{\mathrm{B}bb}{\sqrt{\mathrm{A}} - \mathfrak{B}} + b\sqrt{(\mathrm{A} + 2\mathrm{B}x + \mathrm{C}xx)};$$

vnde casus praecedens manifesto sequitur, si ponatur $\mathrm{B} = 0$.

Verum ope leuis substitutionis hae formulae, vbi adest B, ad illum casum vbi $B = 0$ reduci possunt.

Problema 74.

585. Si $\Pi : z$ significet eam functionem ipsius z, quae oritur ex integratione formulae $\int \frac{\partial z}{\sqrt{(A + Czz)}}$, integrali hoc ita sumto, vt euanescat posito $z = 0$, comparationem inter huiusmodi functiones instituere.

Solutio.

Consideretur haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(A + Cxx)}} = \frac{\partial y}{\sqrt{(A + Cyy)}}$$

vnde cum sit per hypothesin

$$\int \frac{\partial x}{\sqrt{(A + Cxx)}} = \Pi : x \text{ et } \int \frac{\partial y}{\sqrt{(A + Cyy)}} = \Pi : y,$$

vtroque integrali ita sumto, vt euanescat illud posito $x = 0$, hoc vero posito $y = 0$, integrale completum erit

$$\Pi : y = \Pi : x + C.$$

Ante autem vidimus, hoc integrale esse

$$y = x \sqrt{\frac{A + Cbb}{A}} + b \sqrt{\frac{A + Cxx}{A}},$$

vbi posito $x = 0$ fit $y = b$; quare cum $\Pi : 0 = 0$, erit

$$\Pi : y = \Pi : x + \Pi : b;$$

cui ergo aequationi transcendentali satisfacit haec algebraica

$$y = x \sqrt{\frac{A + Cbb}{A}} + b \sqrt{\frac{A + Cxx}{A}}.$$

Simili modo sumto b negatiue, haec aequatio

$$\Pi : y = \Pi : x - \Pi : b$$

conuenit cum hac

$$y = x \sqrt{\frac{A + Cbb}{A}} - b \sqrt{\frac{A + Cxx}{A}},$$

sicque tam summa, quam differentia duarum huiusmodi functionum

ctionum per similem functionem exprimi potest. Hic iam nullo habito discrimine inter quantitates variabiles et constantes, dum $\Pi : z$ functionem determinatam ipsius z significat, scilicet

$$\Pi : z = \int \frac{\partial z}{\sqrt{(A + Czz)}},$$

quae vt assumsimus euanescat posito $z = 0$, vt hoc signandi modo recepto sit

$$\Pi : r = \Pi : p + \Pi : q,$$

debet esse

$$r = p \sqrt{\frac{A + Cqq}{A}} + q \sqrt{\frac{A + Cpp}{A}};$$

vt vero sit

$$\Pi : r = \Pi : p - \Pi : q,$$

debet esse

$$r = p \sqrt{\frac{A + Cqq}{A}} - q \sqrt{\frac{A + Cpp}{A}},$$

vtrinque autem sublata irrationalitate prodit inter p, q, r haec aequatio

$$p^4 + q^4 + r^4 - 2ppqq - 2pprr - 2qqrr = \frac{4Cppqqrr}{A},$$

cuius forma hanc suppeditat proprietatem, vt si p, q, r sint latera cuiusdam trianguli, eique circumscribatur circulus, cuius diameter vocetur $= T$, semper sit $A + CTT = 0$. Illa autem aequatio ob plures quas complectitur radices, satisfacit huic relationi

$$\Pi : p \pm \Pi : q \pm \Pi : r = 0.$$

Corollarium I.

586. Hinc statim deducitur nota arcuum circularium comparatio, ponendo $A = 1$ et $C = -1$. Tum enim fit

$$\Pi : z = \int \frac{\partial z}{\sqrt{(1 - zz)}} = \text{Ang. sin. } z,$$

hincque vt sit

$$\text{Ang. fin. } r = \text{Ang. fin. } p + \text{Ang. fin. } q,$$

oportet esse

$$r = p\surd(1 - qq) + q\surd(1 - pp),$$

et vt sit

$$\text{Ang. fin. } r = \text{Ang. fin. } p - \text{Ang. fin. } q,$$

debet esse

$$r = p\surd(1 - qq) - q\surd(1 - pp),$$

vti constat.

Corollarium 2.

587. Si sit $A = 1$ et $C = 1$, erit

$$\Pi : z = \int \frac{\partial z}{\surd(1 + zz)} = l\,[z + \surd(1 + zz)],$$

vnde vt sit

$$l\,[r + \surd(1 + rr)] = l\,[p + \surd(1 + pp)] + l\,[q + \surd(1 + qq)],$$

erit

$$r = p\surd(1 + qq) + q\surd(1 + pp),$$

vt autem sit

$$l\,[r + \surd(1 + rr)] = l\,[p + \surd(1 + pp)] - l\,[q + \surd(1 + qq)],$$

erit

$$r = p\surd(1 + qq) - q\surd(1 + pp),$$

vti ex indole logarithmorum sponte liquet.

Corollarium 3.

588. Si ponamus in priori formula generali $q = p$, vt sit

$$\Pi : r = 2\,\Pi : p, \text{ erit}$$

$$r = 2p\surd\frac{A + Cpp}{A}.$$

Hinc porro si fiat

$$q = 2p\surd\frac{A + Cpp}{A}, \text{ erit}$$

$\Pi : r$

$$\Pi : r = \Pi : p + 2\,\Pi : p = 3\,\Pi : p,$$

fumto

$$r = p\sqrt{\tfrac{A + Cqq}{A}} + q\sqrt{\tfrac{A + Cpp}{A}}.$$

Eſt vero

$$\sqrt{\tfrac{A + Cqq}{A}} = \sqrt{[1 + \tfrac{4Cpp}{A}(1 + \tfrac{Cpp}{A})]} = 1 + \tfrac{2Cpp}{A},$$

vnde vt ſit

$$\Pi : r = 3\,\Pi : p \text{ fit}$$

$$r = p\,(1 + \tfrac{2Cpp}{A}) + 2p\,(1 + \tfrac{Cpp}{A}) = 3p + \tfrac{4Cp^3}{A}.$$

Scholion.

589. Quo haec multiplicatio facilius continuari queat, praeter relationem aequationi

$$\Pi : r = \Pi : p + \Pi : q$$

reſpondentem, quae eſt

$$r = p\sqrt{\tfrac{A + Cqq}{A}} + q\sqrt{\tfrac{A + Cpp}{A}},$$

notetur aequatio

$$\Pi : p = \Pi : r - \Pi : q,$$

cui reſpondet relatio

$$p = r\sqrt{\tfrac{A + Cqq}{A}} - q\sqrt{\tfrac{A + Crr}{A}};\ \text{vnde fit}$$

$$\sqrt{\tfrac{A + Crr}{A}} = \tfrac{r}{q}\sqrt{\tfrac{A + Cqq}{A}} - \tfrac{p}{q} = \tfrac{p}{q}\left(\tfrac{A + Cqq}{A}\right)$$
$$+ \sqrt{\left(\tfrac{A + Cpp}{A}\right)\left(\tfrac{A + Cqq}{A}\right)} - \tfrac{p}{q},\ \text{ſeu}$$

$$\sqrt{\tfrac{A + Crr}{A}} = \tfrac{Cpq}{A} + \sqrt{\left(\tfrac{A + Cpp}{A}\right)\left(\tfrac{A + Cqq}{A}\right)}.$$

Quare vt ſit

$$\Pi : r = \Pi : p + \Pi : q,$$

habemus non ſolum

$$r = p\sqrt{(1 + \tfrac{C}{A}qq)} + q\sqrt{(1 + \tfrac{C}{A}pp)},$$

ſed etiam

$\sqrt{}$

$$\sqrt{(1+\tfrac{C}{A}rr)} = \tfrac{C}{A}pq + \sqrt{(1+\tfrac{C}{A}pp)(1+\tfrac{C}{A}qq)}.$$

Ponamus breuitatis gratia $\sqrt{(1+\frac{C}{A}pp)} = P$, et fumto $q = p$ vt fit

$$\Pi : r = 2\Pi : p, \text{ erit}$$
$$r = 2Pp \text{ et } \sqrt{(1+\tfrac{C}{A}rr)} = \tfrac{C}{A}pp + PP,$$

qui valor ipfius r pro q fumtus dabit

$$\Pi : r = 3\Pi : p,$$

exiftente

$$r = \tfrac{C}{A}p^3 + 3PPp, \text{ et}$$
$$\sqrt{(1+\tfrac{C}{A}rr)} = \tfrac{3C}{A}Ppp + P^3.$$

Hic valor ipfius r denuo pro q fumtus, dabit

$$\Pi : r = 4\Pi : p,$$

exiftente

$$r = \tfrac{4C}{A}Pp^3 + 4P^3p, \text{ et}$$
$$\sqrt{(1+\tfrac{C}{A}rr)} = \tfrac{CC}{AA}p^4 + \tfrac{6C}{A}PPpp + P^4.$$

Loco q fubftituatur hic valor ipfius r, vt prodeat

$$\Pi : r = 5\Pi : p,$$

exiftente

$$r = \tfrac{CC}{AA}p^5 + \tfrac{10C}{A}PPp^3 + 5P^4p, \text{ et}$$
$$\sqrt{(1+\tfrac{C}{A}rr)} = \tfrac{5CC}{AA}Pp^4 + \tfrac{10C}{A}P^3pp + P^5.$$

Atque hinc generatim concludere licet, vt fit

$$\Pi : r = n\Pi : p,$$

effe debere

$$r\sqrt{\tfrac{C}{A}} = \tfrac{1}{2}(P + p\sqrt{\tfrac{C}{A}})^n - \tfrac{1}{2}(P - p\sqrt{\tfrac{C}{A}})^n \text{ et}$$
$$\sqrt{(1+\tfrac{C}{A}rr)} = \tfrac{1}{2}(P + p\sqrt{\tfrac{C}{A}})^n + \tfrac{1}{2}(P - p\sqrt{\tfrac{C}{A}})^n, \text{ feu}$$
$$r = \tfrac{\sqrt{A}}{2\sqrt{C}}(P + p\sqrt{\tfrac{C}{A}})^n - \tfrac{\sqrt{A}}{2\sqrt{C}}(P - p\sqrt{\tfrac{C}{A}})^n.$$

Haec

Haec igitur relatio inter p et r satisfaciet huic aequationi differentiali

$$\frac{\partial r}{\sqrt{(A+Crr)}}=\frac{n\,\partial p}{\sqrt{(A+Cpp)}},$$

dum meminerimus esse $P=\sqrt{(1+\frac{Cpp}{A})}$.

Problema 75.

590. Si ponatur $\int\frac{\partial z}{\sqrt{(A+Czz)}}=\Pi:z$, integrali ita sumto vt euanescat posito $z=f$, vnde $\Pi:z$ sit functio determinata ipsius z, comparationem inter huiusmodi functiones instituere.

Solutio.

Consideretur haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(A+Cxx)}}+\frac{\partial y}{\sqrt{(A+Cyy)}}=0,$$

vnde integrando fit

$$\Pi:x+\Pi:y=\text{Const.}$$

Integrale autem fit quoque

$$\alpha+\gamma(xx+yy)+2\delta xy=0,$$

quod vt locum habeat necesse est, sit

$$-\alpha\gamma=Am,\ \text{et}\ \delta\delta-\gamma\gamma=Cm:$$

tum vero erit

$$\gamma x+\delta y=\sqrt{m}(A+Cyy),\ \text{et}\ \gamma y+\delta x=\sqrt{m}(A+Cxx).$$

Ponamus constantem integratione ingressam ita definiri, vt posito $x=a$ fiat $y=b$, et integrale erit

$$\Pi:x+\Pi:y=\Pi:a+\Pi:b.$$

Pro forma autem algebraica inuenienda, sit breuitatis gratia

$$\sqrt{(A+Caa)}=\mathfrak{A}\ \text{et}\ \sqrt{(A+Cbb)}=\mathfrak{B},$$

eritque

$$\gamma a + \delta b = \mathfrak{B}\surd m \text{ et } \gamma b + \delta a = \mathfrak{A}\surd m;$$

vnde colligitur

$$\gamma = \frac{\mathfrak{A}b - \mathfrak{B}a}{bb - aa}\surd m \text{ et } \delta = \frac{\mathfrak{B}b - \mathfrak{A}a}{bb - aa}\surd m.$$

Quocirca aequatio integralis algebraica erit

$$(\mathfrak{A}b - \mathfrak{B}a)x + (\mathfrak{B}b - \mathfrak{A}a)y = (bb - aa)\surd(\mathrm{A} + \mathrm{C}yy)$$

seu

$$(\mathfrak{A}b - \mathfrak{B}a)y + (\mathfrak{B}b - \mathfrak{A}a)x = (bb - aa)\surd(\mathrm{A} + \mathrm{C}xx).$$

Hinc y per x ita definitur, vt sit

$$y = \frac{(\mathfrak{A}a - \mathfrak{B}b)x + (bb - aa)\surd(\mathrm{A} + \mathrm{C}xx)}{\mathfrak{A}b - \mathfrak{B}a},$$

quae fractio supra et infra per $\mathfrak{A}b + \mathfrak{B}a$ multiplicando, ob

$$\mathfrak{A}\mathfrak{A}bb - \mathfrak{B}\mathfrak{B}aa = \mathrm{A}(bb - aa) \text{ et}$$

$$(\mathfrak{A}a - \mathfrak{B}b)(\mathfrak{A}b + \mathfrak{B}a) = (\mathfrak{A}\mathfrak{A} - \mathfrak{B}\mathfrak{B})ab - \mathfrak{A}\mathfrak{B}(bb - aa) = - (bb - aa)(\mathrm{C}ab + \mathfrak{A}\mathfrak{B}),$$

abit in

$$y = -\frac{(\mathrm{C}ab + \mathfrak{A}\mathfrak{B})x}{\mathrm{A}} + \frac{(\mathfrak{A}b + \mathfrak{B}a)\surd(\mathrm{A} + \mathrm{C}xx)}{\mathrm{A}}.$$

Hinc porro colligitur

$$(bb - aa)\surd(\mathrm{A} + \mathrm{C}yy) = (\mathfrak{A}b - \mathfrak{B}a)x - \frac{(\mathfrak{B}b - \mathfrak{A}a)^2 x}{\mathfrak{A}b - \mathfrak{B}a} + \frac{(\mathfrak{B}b - \mathfrak{A}a)(bb - aa)}{\mathfrak{A}b - \mathfrak{B}a}\surd(\mathrm{A} + \mathrm{C}xx);$$

seu

$$\surd(\mathrm{A} + \mathrm{C}yy) = -\frac{\mathrm{C}(bb - aa)}{\mathfrak{A}b - \mathfrak{B}a}x + \frac{\mathfrak{B}b - \mathfrak{A}a}{\mathfrak{A}b - \mathfrak{B}a}\surd(\mathrm{A} + \mathrm{C}xx);$$

vbi iterum supra et infra multiplicando per $\mathfrak{A}b + \mathfrak{B}a$, fit

$$\surd(\mathrm{A} + \mathrm{C}yy) = -\frac{\mathrm{C}(\mathfrak{A}b + \mathfrak{B}a)}{\mathrm{A}}x + \frac{(\mathrm{C}ab + \mathfrak{A}\mathfrak{B})}{\mathrm{A}}\surd(\mathrm{A} + \mathrm{C}xx).$$

Necesse autem est valorem formulae $\surd(\mathrm{A} + \mathrm{C}yy)$ hoc modo potius definiri quam extractione radicis, qua ambiguitas implicaretur. Quocirca haec aequatio transcendens

$$\Pi : r + \Pi : s = \Pi : p + \Pi : q$$

praebet sequentem determinationem algebraicam, si quidem breui-

breuitatis gratia ponamus $\sqrt{(A + Cpp)} = P$, $\sqrt{(A + Cqq)} = Q$ et $\sqrt{(A + Crr)} = R$, fcilicet vt fit

$$\Pi : s = \Pi : p + \Pi : q - \Pi : r, \text{ erit}$$

$$s = \frac{-PQr - Cpqr + PRq + QRp}{A} \text{ et}$$

$$\sqrt{(A + Css)} = \frac{-CPqr - CQpr + CRpq + PQR}{A}, \text{ feu}$$

$$\sqrt{(A + Css)} = \frac{PQR + C(Rpq - Pqr - Qpr)}{A}.$$

Corollarium 1.

591. Quoniam eft per hypothefin $\Pi : f = 0$, fi ponamus breuitatis gratia $\sqrt{(A + Cff)} = F$, et $r = f$, vt fit $R = F$, haec aequatio

$$\Pi : s = \Pi : p + \Pi : q$$

praebet

$$s = \frac{F(Pq + Qp) - PQf - Cfpq}{A}, \text{ et}$$

$$\sqrt{(A + Css)} = \frac{FPQ + CFpq - Cf(Pq + Qp)}{A}.$$

Corollarium 2.

592. Si ponamus $q = f$ et $Q = F$, vt fit $\Pi : q = 0$, haec aequatio

$$\Pi : s = \Pi : p - \Pi : r$$

praebet

$$s = \frac{F(Rp - Pr) + fPR - Cfpr}{A} \text{ et}$$

$$\sqrt{(A + Css)} = \frac{FPR - CFpr + Cf(Rp - Pr)}{A}.$$

Corollarium 3.

593. Si fit $C = 0$ et $A = 1$, erit

$$\Pi : z = \int \partial z = z - f,$$

quia integrale ita capi debet, vt euanefcat pofito $z = f$. Tum ergo erit $P = 1$, $Q = 1$ et $R = 1$; vnde vt fit

 $\Pi : s$

$$\Pi : s = \Pi : p + \Pi : q - \Pi : r,$$

seu $s = p + q - r$, oportet esse

$$s = -r + q + p \text{ et } \sqrt{(1 + 0ss)} = 1,$$

vti per se constat.

Corollarium 4.

594. Si sumatur $A = 1$ et $C = -1$, fiatque $\Pi : z = \text{Ang. cos. } z$, vt sit $f = 1$, erit

$$\text{Arc. cos. } s = \text{Arc. cos. } p + \text{Arc. cos. } q - \text{Arc. cos. } r,$$

si fuerit

$$s = pqr - PQr + PRq + QRp \text{ et}$$
$$\sqrt{(1 - ss)} = PQR + Pqr + Qpr - Rpq;$$

vnde sumto $r = 1$, vt sit $R = 0$, et $\text{Arc. cos. } r = 0$, erit $s = pq - PQ$ et $\sqrt{(1 - ss)} = Pq + Qp$.

Scholion.

595. Hinc notae regulae pro cosinibus deducuntur; quas fusius non prosequor. Verum casus facillimus, quo $A = 0$ et $C = 1$, hincque fit $\Pi : z = \int \frac{\partial z}{z} = lz$, existente $f = 1$, insigni difficultate premi videtur, ob expressiones pro s et $\sqrt{(A + Czz)} = z$ in infinitum abeuntes. Cui incommodo vt occurratur, primo quidem numerus A vt infinite paruus spectetur, eritque

$$P = \sqrt{(pp + A)} = p + \frac{A}{2p},\quad Q = q + \frac{A}{2q},\quad R = r + \frac{A}{2r}.$$

Quare vt fiat $ls = lp + lq - lr$, reperitur

$$As = -r\left(p + \frac{A}{2p}\right)\left(q + \frac{A}{2q}\right) - pqr$$
$$+ q\left(p + \frac{A}{2p}\right)\left(r + \frac{A}{2r}\right) + p\left(q + \frac{A}{2q}\right)\left(r + \frac{A}{2r}\right);$$

ac singulis membris euolutis

As

$$A s = -\frac{Aqr}{2p} - \frac{Apr}{2q} + \frac{Aqr}{2p} + \frac{Apq}{2r} + \frac{Apr}{2q} + \frac{Apq}{2r}$$

seu $s = \frac{pq}{r}$, vti natura logarithmorum exigit. Caeterum ex formulis inuentis haud difficulter multiplicatio huiusmodi functionum transcendentium colligitur, veluti vt sit $\Pi : y = n\Pi : x$, relatio inter x et y algebraice assignari poterit.

Problema 76.

596. Si ponatur $\Pi : z = \int \frac{\partial z (L + Mzz)}{\sqrt{(A + Czz)}}$, sumto hoc integrali ita vt euanescat posito $z = 0$, comparationem inter huiusmodi functiones transcendentes inuestigare.

Solutio.

Statuatur inter binas variabiles x et y ista relatio

$$\alpha + \gamma (xx + yy) + 2\delta xy = 0,$$

vnde fit

$$y = \frac{-\delta x + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma) xx]}}{\gamma}.$$

Ponatur $-\alpha\gamma = A m$ et $\delta\delta - \gamma\gamma = C m$, vt sit

$$\gamma y + \delta x = \sqrt{m (A + C xx)} \text{ et}$$

$$\gamma x + \delta y = \sqrt{m (A + C yy)}.$$

At illam aequationem differentiando fit

$$\partial x (\gamma x + \delta y) + \partial y (\gamma y + \delta x) = 0, \text{ seu}$$

$$\frac{\partial x}{\sqrt{(A + C xx)}} + \frac{\partial y}{\sqrt{(A + C yy)}} = 0.$$

Iam statuatur

$$\frac{\partial x (L + M xx)}{\sqrt{(A + C xx)}} + \frac{\partial y (L + M yy)}{\sqrt{(A + C yy)}} = \partial V \sqrt{m},$$

vt sit integrando

$$\Pi : x + \Pi : y = \text{Const.} + V \sqrt{m}.$$

Cum igitur sit

$$\frac{\partial y}{\sqrt{(A+Cyy)}} = \frac{-\partial x}{\sqrt{(A+Cxx)}}, \text{ erit}$$
$$\partial V \sqrt{m} = \frac{M\partial x(xx-yy)}{\gamma(A+Cxx)},$$
hincque ob
$$y = \frac{\sqrt{m(A+Cxx)} - \delta x}{\gamma}, \text{ erit}$$
$$xx - yy = \frac{1}{\gamma\gamma}(\gamma\gamma xx - mA - mCxx - \delta\delta xx + 2\delta x\sqrt{m(A+Cxx)}).$$
At $\gamma\gamma - \delta\delta = -mC$, ergo
$$\partial V \sqrt{m} = \frac{M\partial x(2\delta x\sqrt{m(A+Cxx)} - mA - 2mCxx)}{\gamma\gamma\sqrt{(A+Cxx)}},$$
cuius integrale commode capi poteſt, dum ſit
$$V\sqrt{m} = \frac{\delta Mxx\sqrt{m}}{\gamma\gamma} - \frac{Mmx}{\gamma\gamma}\sqrt{(A+Cxx)},$$
quae formula ob
$$\sqrt{m(A+Cxx)} = \gamma y + \delta x; \text{ abit in}$$
$$V\sqrt{m} = \frac{\delta Mxx - \gamma Mxy - \delta Mxx}{\gamma\gamma}\sqrt{m} = -\frac{Mxy}{\gamma}\sqrt{m}.$$
Quocirca habebimus
$$\Pi : x + \Pi : y = \text{Conſt.} - \frac{Mxy}{\gamma}\sqrt{m},$$
exiſtente
$$\gamma y + \delta x = \sqrt{m(A+Cxx)} \text{ et } \gamma x + \delta y = \sqrt{m(A+Cyy)},$$
ac praeterea
$$-\alpha\gamma = Am \text{ et } \delta\delta - \gamma\gamma = Cm.$$
Ad conſtantem definiendam ſumamus, poſito $x = 0$ fieri $y = b$, vt ſit
$$\Pi : x + \Pi : y = \Pi : b - \frac{Mxy}{\gamma}\sqrt{m}.$$
Tum vero eſt
$$\gamma b = \sqrt{mA} \text{ et } \delta b = \sqrt{(mA + mCbb)},$$
ergo
$$\gamma = \frac{\sqrt{mA}}{b} \text{ et } \delta = \frac{\sqrt{(mA+mCbb)}}{b}.$$
Hinc ergo concludimus, ſi fuerit

$\gamma\sqrt{A}$

$$y\sqrt{A}+x\sqrt{(A+Cbb)}=b\sqrt{(A+Cxx)},$$

et quod eodem redit

$$x\sqrt{A}+y\sqrt{(A+Cbb)}=b\sqrt{(A+Cyy)}, \text{ fore}$$

$$\Pi:x+\Pi:y=\Pi:b-\frac{Mbxy}{\sqrt{A}}:$$

denotante Π eiusmodi functionem quantitatis suffixae, vt sit

$$\Pi:z=\int\frac{\partial z(L+Mzz)}{\sqrt{(A+Czz)}},$$

integrali hoc ita sumto, vt euanescat posito $z=0$. Natura harum functionum stabilita, ac sublato discrimine inter quantitates constantes ac variabiles, erit

$$\Pi:r=\Pi:p+\Pi:q+\frac{Mpqr}{\sqrt{A}},$$

si fuerit

$$q\sqrt{A}+p\sqrt{(A+Crr)}=r\sqrt{(A+Cpp)} \text{ et}$$

$$p\sqrt{A}+q\sqrt{(A+Crr)}=r\sqrt{(A+Cqq)}$$

vnde fit

$$r=\frac{p\sqrt{(A+Cqq)}+q\sqrt{(A+Cpp)}}{\sqrt{A}} \text{ et}$$

$$\sqrt{(A+Crr)}=\frac{Cpq+\sqrt{(A+Cpp)(A+Cqq)}}{\sqrt{A}}.$$

Corollarium 1.

597. Sumto z negatiuo est

$$\Pi:-z=-\Pi:z,$$

vnde capiendo quantitates p et q negatiue, fiet

$$\Pi:p+\Pi:q+\Pi:r=\frac{Mpqr}{\sqrt{A}},$$

si fuerit

$$p\sqrt{A}+q\sqrt{(A+Crr)}+r\sqrt{(A+Cqq)}=0 \text{ seu}$$

$$q\sqrt{A}+p\sqrt{(A+Crr)}+r\sqrt{(A+Cpp)}=0 \text{ seu}$$

$$r\sqrt{A}+p\sqrt{(A+Cqq)}+q\sqrt{(A+Cpp)}=0 \text{ vel}$$

$$Cpq-\sqrt{A(A+Crr)}+\sqrt{(A+Cpp)(A+Cqq)}=0$$

ex

ex qua formatur haec relatio

$$Cpqr + p\sqrt{(A+Cqq)(A+Crr)} + q\sqrt{(A+Cpp)(A+Crr)} + r\sqrt{(A+Cpp)(A+Cqq)} = 0.$$

Corollarium 2.

598. Hac ergo methodo tres huiusmodi functiones $\Pi : z$ exhiberi possunt, quarum summam algebraice exprimere licet; quod autem de summa ostendimus, valet quoque de summa binarum demta tertia.

Corollarium 3.

599. Si ponamus $L = A$ et $M = C$, functio proposita $\Pi : z = \int \partial z \sqrt{(A + Czz)}$, exprimit aream curuae, cuius abscissae z conuenit applicata $\sqrt{(A + Czz)}$; et summa trium huiusmodi arearum ita algebraice dabitur:

$$\Pi : p + \Pi : q + \Pi : r = \frac{Cpqr}{\sqrt{A}}$$

si inter p, q, r superior relatio statuatur.

Scholion.

600. Haec proprietas inde est nata, quod differentiale ∂V integrationem admisit. Cum nempe esset

$$\partial V \sqrt{m} = \frac{M\partial x(xx - yy)}{\sqrt{(A + Cxx)}},\ \text{ob}$$

$$\sqrt{m}(A + Cxx) = \gamma y + \delta x,\ \text{erit}$$

$$\partial V = \frac{M\partial x(xx - yy)}{\gamma y + \delta x},$$

cuius integrale commode ex aequatione assumta

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0$$

definiri potest. Ponatur enim

$$xx + yy = tt \text{ et } xy = u,\ \text{erit}$$

$$\alpha + \gamma tt + 2\delta u = 0$$

et

et differentialibus fumendis

$$x\partial x + y\partial y = t\partial t;\quad x\partial y + y\partial x = \partial u \text{ et } \gamma t\partial t + \delta\partial u = 0;$$

ex binis prioribus colligitur

$$(xx - yy)\partial x = xt\partial t - y\partial u, \text{ et ob } t\partial t = -\tfrac{\delta\partial u}{\gamma}, \text{ erit}$$

$$(xx - yy)\partial x = -\tfrac{\partial u}{\gamma}(\delta x + \gamma y),$$

ita vt fit

$$\tfrac{\partial x(xx - yy)}{\gamma y + \delta x} = -\tfrac{\partial u}{\gamma}, \text{ hincque } \partial V = -\tfrac{M\partial u}{\gamma},$$

vnde manifefto fequitur

$$V = -\tfrac{Mu}{\gamma} = -\tfrac{Mxy}{\gamma},$$

vti in folutione operofius eruimus. Verum hac operatione commode vti licebit in fequente problemate, vbi formulas magis complexas fumus contemplaturi.

Problema 77.

601. Si ponatur

$$\Pi : z = \int \tfrac{\partial z(1 + Mz^2 + Nz^4 + Oz^6 + etc.)}{\sqrt{(A + Czz)}},$$

integrali hoc ita fumto vt euanefcat pofito $z = 0$, comparationem inter huiusmodi functiones tranfcendentes inueftigare.

Solutio.

Pofita vt ante inter variabiles x et y hac relatione

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0,$$

fit

$$-\alpha\gamma = Am \text{ et } \delta\delta - \gamma\gamma = Cm,$$

fietque

$$\gamma y + \delta x = \sqrt{m}(A + Cxx) \text{ et } \gamma x + \delta y = \sqrt{m}(A + Cyy),$$

fumtisque differentialibus

$$\frac{\partial x}{\sqrt{(A+Cxx)}}+\frac{\partial y}{\sqrt{(A+Cyy)}}=0.$$

Iam ſtatuatur

$$\frac{\partial x(L+Mx^2+Nx^4+Ox^6)}{\sqrt{(A+Cxx)}}+\frac{\partial y(L+Myy+Ny^4+Oy^6)}{\sqrt{(A+Cyy)}}=\partial V\sqrt{m}$$

vt ſit

$$\Pi:x+\Pi:y=\text{Conſt.}+V\sqrt{m}.$$

At ob $\frac{\partial y}{\sqrt{(A+Cyy)}}=-\frac{\partial x}{\sqrt{(A+Cxx)}}$, iſta aequatio abit in

$$\frac{\partial x[M(xx-yy)+N(x^4-y^4)+O(x^6-y^6)]}{\sqrt{(A+Cxx)}}=\partial V\sqrt{m},$$

et ob $\sqrt{m}(A+Cxx)=\gamma y+\delta x$, in hanc

$$\frac{\partial x(xx-yy)[M+N(xx+yy)+O(x^4+xxyy+y^4)]}{\gamma y+\delta x}=\partial V.$$

Sit nunc $xx+yy=tt$ et $xy=u$, vt habeatur

$$\alpha+\gamma tt+\delta u=0 \text{ et } \gamma t\partial t+\delta\partial u=0,$$

ſeu $t\partial t=-\frac{\delta\partial u}{\gamma}$,

atque ob

$$x\partial x+y\partial y=t\partial t \text{ et } x\partial y+y\partial x=\partial u$$

in colligimus

$$(xx-yy)\partial x=xt\partial t-y\partial u=-\frac{\partial u}{\gamma}(\gamma y+\delta x),$$

ideoque

$$\frac{\partial x(xx-yy)}{\gamma y+\delta x}=-\frac{\partial u}{\gamma},$$

vnde habebimus

$$\partial V=-\frac{\partial u}{\gamma}[M+N(xx+yy)+O(x^4+xxyy+y^4)].$$

At eſt

$$xx+yy=tt=\frac{-\alpha-\delta u}{\gamma} \text{ et}$$

$$x^4+xxyy+y^4=t^4-uu.$$

Notetur autem eſſe $\frac{\partial u}{\gamma}=-\frac{t\partial t}{\delta}$, vnde concludimus

$$\partial V=-\frac{M\partial u}{\gamma}+\frac{Nt^3\partial t}{\delta}+\frac{Ot^5\partial t}{\delta}+\frac{Ouu\partial u}{\gamma},$$

ſicque

ficque prodit integrando

$$V = -\frac{Mu}{\gamma} + \frac{Nt^4}{4\delta} + \frac{Ot^6}{6\delta} + \frac{Ou^3}{3\gamma}.$$

Quodfi iam ponamus fieri $y = b$ fi $x = 0$, erit $\gamma = \frac{\sqrt{mA}}{b}$, $\delta = \frac{\sqrt{m(A + Cbb)}}{b}$, et $a = -b\sqrt{mA}$, tum vero

$$y\sqrt{A} + x\sqrt{(A + Cbb)} = b\sqrt{(A + Cxx)}$$
$$x\sqrt{A} + y\sqrt{(A + Cbb)} = b\sqrt{(A + Cyy)} \text{ et}$$
$$b\sqrt{A} = x\sqrt{(A + Cyy)} + y\sqrt{(A + Cxx)}.$$

Hinc cum fit

$$V = -\frac{Mbxy}{\sqrt{mA}} + \frac{Nb(xx + yy)^2}{4\sqrt{m(A + Cbb)}} + \frac{Ob(xx + yy)^3}{6\sqrt{m(A - Cbb)}} + \frac{Obx^3y^3}{3\sqrt{mA}},$$

noftra relatio, cui fatisfaciunt praecedentes determinationes, inter functiones tranfcendentes, erit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} + \frac{Nb(xx + yy)^2}{4\sqrt{(A + Cbb)}} + \frac{Ob(xx + yy)^3}{6\sqrt{(A + Cbb)}}$$
$$+ \frac{Obx^3y^3}{3\sqrt{A}} - \frac{Nb^5}{4\sqrt{(A + Cbb)}} - \frac{Ob^7}{6\sqrt{(A + Cbb)}}:$$

vbi notandum eft effe in rationalibus

$$-b\sqrt{A} + \frac{(xx + yy)\sqrt{A}}{b} + \frac{2xy\sqrt{(A + Cbb)}}{b} = 0, \text{ feu}$$
$$xx + yy = bb - \frac{2xy\sqrt{(A + Cbb)}}{\sqrt{A}}.$$

Hinc colligitur

$$(xx + yy)^2 - b^4 = -\frac{4bbxy\sqrt{(A + Cbb)}}{\sqrt{A}} + \frac{4xxyy(A + Cbb)}{A} \text{ et}$$
$$(xx + yy)^3 - b^6 = -\frac{6b^4xy\sqrt{(A + Cbb)}}{\sqrt{A}} + \frac{12bbxxyy(A + Cbb)}{A}$$
$$-\frac{8x^3y^3(A + Cbb)^{\frac{3}{2}}}{A\sqrt{A}};$$

ita vt noftra aequatio fit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} - \frac{Nb^3xy}{\sqrt{A}} + \frac{Nbxxyy}{A}\sqrt{(A + Cbb)}$$
$$-\frac{Ob^5xy}{\sqrt{A}} + \frac{2Ob^3xxyy}{A}\sqrt{(A + Cbb)} - \frac{Obx^3y^3}{3A\sqrt{A}}(3A + 4Cbb)$$

Corol-

Corollarium 1.

602. Si ponamus $b = r$, $x = -p$, $y = -q$, erit noſtra aequatio

$$\Pi : p + \Pi : q + \Pi : r = \frac{pqr}{\sqrt{A}}(M + Nrr + Or^4)$$
$$- \frac{ppqq\sqrt{(A + Crr)}}{A}(Nr + 2Or^3) + \frac{Op^3q^3r}{3A\sqrt{A}}(3A + 4Crr),$$

exiſtente $pp + qq = rr - \frac{2pq}{\sqrt{A}}\sqrt{(A + Crr)}$, vnde fit

$$\frac{\sqrt{(A + Crr)}}{\sqrt{A}} = \frac{rr - pp - qq}{2pq}.$$

Corollarium 2.

603. Subſtituto hoc valore pro $\frac{\sqrt{(A + Crr)}}{\sqrt{A}}$, ſequens obtinebitur aequatio, in quam ternae quantitates p, q, r aequaliter ingrediuntur

$$\Pi : p + \Pi : q + \Pi : r = \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2\sqrt{A}}(pp + qq + rr)$$
$$+ \frac{Opqr}{3\sqrt{A}}(p^4 + q^4 + r^4 + ppqq + pprr + qqrr)$$

cui ſatisfaciunt formulae ſupra datae, vel haec rationalis

$$\frac{4Cppqqrr}{A} = p^4 + q^4 + r^4 - 2ppqq - 2pprr - 2qqrr.$$

Corollarium 3.

604. Si numeratori formulae integralis adhuc adieciſſemus terminum Pz^8, vt eſſet

$$\Pi : z = \int \frac{\partial z(L + Mz^2 + Nz^4 + Oz^6 + Pz^8)}{\sqrt{(A + Czz)}},$$

ad aequationem modo inuentam adhuc acceſſiſſet terminus

$$\frac{Ppqr}{4\sqrt{A}}(p^6 + q^6 + r^6 + ppq^4 + ppr^4 + p^4qq + p^4rr + q^4rr + qqr^4 + \tfrac{2}{3}ppqqrr)$$

Scholion.

605. Iſtae relationes quoque ex ſuperioribus reductionibus deriuari poſſunt, cum enim inde fit $\Pi : z = E\int\frac{\partial z}{\sqrt{(A + Czz)}}$ + *quantitate algebraica*, ſi hic pro z ſucceſſiue quantitates p,

q,

q, r fubftituamus, ita a fe inuicem pendentes, vt ante declarauimus, erit

$$\int \frac{\partial p}{\sqrt{(A+Cpp)}} + \int \frac{\partial q}{\sqrt{(A+Cqq)}} + \int \frac{\partial r}{\sqrt{(A+Crr)}} = 0;$$

vnde concludimus

$$\Pi : p + \Pi : q + \Pi : r = f : p + f : q + f : r,$$

denotante f functionem quandam algebraicam quantitatis fuffixae: atque fumma harum trium functionum rediret ad expresfionem ante inuentam, fi modo relationis inter p, q, r datae ratio habeatur: fcilicet inde littera C eliminari deberet. Haec autem reductio ingentem laborem requireret. Hic vero imprimis methodum, qua hic fum vfus, fpectari conuenit, quae cum fit prorfus fingularis, ad magis arduam deducere videtur. Certe comparatio functionum tranfcendentium, quam in capite fequente fum traditurus, vix alia methodo inueftigari poffe videtur, vnde huius methodi vtilitas in fequenti capite potiffimum cernetur.

CAPVT VI.

DE COMPARATIONE QVANTITATVM TRANSCENDENTIVM CONTENTARVM IN FORMA

$$\int \frac{P\,\partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$$

Problema 78.

606.

Proposita relatione inter x et y hac

$$\alpha + \gamma(xx + yy) + 2\delta xy + \zeta xxyy = 0,$$

inde elicere fnnctiones transcendentes formae praescriptae, quas inter se compaiare liceat.

Solutio.

Ex proposita aequatione definiatur vtraque variabilis

$$y = \frac{-\delta x + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4]}}{\gamma + \zeta xx} \text{ et}$$

$$x = \frac{-\delta y + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4]}}{\gamma + \zeta yy},$$

quae radicalia ad formam praescriptam reuocentur ponendo

$$-\alpha\gamma = Am, \quad \delta\delta - \gamma\gamma - \alpha\zeta = Cm \text{ et } -\gamma\zeta = Em;$$

vnde fit

$$\alpha = -\frac{Am}{\gamma}, \quad \zeta = -\frac{Em}{\gamma} \text{ et } \delta\delta = Cm + \gamma\gamma + \frac{AEmm}{\gamma\gamma}.$$

Erit ergo

$$\gamma y + \delta x + \zeta xxy = \sqrt{m(A + Cxx + Ex^4)}$$

$$\gamma x + \delta y + \zeta xyy = \sqrt{m(A + Cyy + Ey^4)}.$$

Ipsa autem aequatio proposita, si differentietur, dat

∂x

$$\partial x\,(\gamma x + \delta y + \zeta xyy) + \partial y\,(\gamma y + \delta x + \zeta xxy) = 0$$

vbi illi valores substituti praebent

$$\frac{\partial x}{\sqrt{(A + Cxx + Ex^4)}} + \frac{\partial y}{\sqrt{(A + Cyy + Ey^4)}} = 0.$$

Viciſſim ergo propoſita hac aequatione differentiali, ei ſatisfaciet haec aequatio finita

$$-Am + \gamma\gamma\,(xx + yy) + 2xy\sqrt{(\gamma^4 + Cm\gamma\gamma + AEmm)}$$
$$- Emxxyy = 0,$$

ſeu ponendo $\frac{\gamma\gamma}{m} = k$, haec

$$-A + k\,(xx + yy) + 2xy\sqrt{(kk + kC + AE)} - Exxyy = 0,$$

quae cum inuoluat conſtantem k, in aequatione differentiali non contentam, ſimul erit integrale completum. Hinc autem fit

$$ky + x\sqrt{(kk + kC + AE)} - Exxy = \sqrt{k(A + Cxx + Ex^4)} \text{ et}$$
$$kx + y\sqrt{(kk + kC + AE)} - Exyy = \sqrt{k(A + Cyy + Ey^4)}.$$

Corollarium 1.

607. Conſtans k ita aſſumi poteſt, vt poſito $x = 0$, fiat $y = b$, oritur autem

$$bk = \sqrt{Ak} \text{ et } b\sqrt{(kk + kC + AE)} = \sqrt{k(A + Cbb + Eb^4)},$$

ergo

$$k = \frac{A}{bb} \text{ et } \sqrt{(kk + kC + AE)} = \frac{1}{bb}\sqrt{A(A + Cbb + Eb^4)},$$

ideoque habebimus

$$Ay + x\sqrt{A(A + Cbb + Eb^4)} - Ebbxxy = b\sqrt{A(A + Cxx + Ex^4)}$$

et

$$Ax + y\sqrt{A(A + Cbb + Eb^4)} - Ebbxyy = b\sqrt{A(A + Cyy + Ey^4)}.$$

Corollarium 2.

608. Haec igitur relatio finita inter x et y erit integrale completum aequationis differentialis

∂x

$$\frac{\partial x}{\sqrt{(A+Cxx+Ex^4)}} + \frac{\partial y}{\sqrt{(A+Cyy+Ey^4)}} = 0,$$

quod rationaliter inter x et y expressum erit

$$A(xx+yy-bb)+2xy\sqrt{A(A+Cbb+Eb^4)}-Ebbxxyy=0.$$

Corollarium. 3.

609. Hinc ergo y ita per x exprimetur, vt sit

$$y=\frac{b\sqrt{A(A+Cxx+Ex^4)}-x\sqrt{A(A+Cbb+Eb^4)}}{A-Ebbxx},$$

atque ex hoc valore elicitur

$$\sqrt{\frac{A+Cyy+Ey^4}{A}}=\frac{(A+Ebbxx)\sqrt{(A+Cbb+Eb^4)(A+Cxx+Ex^4)}-2Ebx(bb+xx)-Cbx(A+Ebbxx)}{(A-Ebbxx)^2}.$$

Corollarium 4.

610. Hinc constantem b pro lubitu determinando infinita integralia particularia exhiberi possunt, quorum praecipua sunt: 1) sumendo $b=0$, vnde fit $y=-x$; 2) sumendo $b=\infty$, vnde fit $y=\frac{\sqrt{A}}{x\sqrt{E}}$. 3) Si $A+Cbb+Eb^4=0$, hincque $bb=\frac{-C+\sqrt{(CC-4AE)}}{2E}$, vnde fit $y=\frac{b\sqrt{A(A+Cxx+Ex^4)}}{A-Ebbxx}$.

Scholion.

611. Hic iam vsus istius methodi, qua retrogrediendo ab aequatione finita ad aequationem differentialem peruenimus, luculenter perspicitur. Cum enim integratio formulae $\int\frac{\partial x}{\sqrt{(A+Cxx+Ex^4)}}$ nullo modo neque per logarithmos neque per arcus circulares perfici posset, mirum sane est talem aequationem differentialem adeo algebraice integrari posse; quae quidem in praecedente capite ope eiusdem methodi sunt tradita, etiam methodo ordinaria erui possunt, dum singulae formulae differentiales vel per logarithmos vel arcus circulares exprimuntur, quorum deinceps comparatio ad aequationem algebraicam reducitur. Verum quia hic talis integratio plane non locum inuenit, nulla certe alia methodus patet, qua idem integrale,

grale, quod hic exhibuimus, inuestigari posset. Quare hoc argumentum diligentius euoluamus.

Problema 79.

612. Si $\Pi:z$ denotet eiusmodi functionem ipsius z, vt sit $\Pi:z=\int\frac{\partial z}{\sqrt{(A+Czz+Ez^4)}}$, integrali ita sumto vt euanescat posito $z=0$, comparationem inter huiusmodi functiones inuestigare.

Solutio.

Posita inter binas variabiles x et y relatione supra definita, vidimus fore

$$\frac{\partial x}{\sqrt{(A+Cxx+Ex^4)}}+\frac{\partial y}{\sqrt{(A+Cyy+Ey^4)}}=0.$$

Hinc cum posito $x=0$ fiat $y=b$, elicitur integrando

$$\Pi:x+\Pi:y=\Pi:b.$$

Cum iam nullum amplius discrimen inter variabiles x, y et constantem b intercedat, statuamus $x=p$, $y=q$, et $b=-r$, vt sit $\Pi:b=-\Pi:r$, atque haec relatio inter functiones transcendentes

$$\Pi:p+\Pi:q+\Pi:r=0$$

per sequentes formulas algebraicas exprimetur,

$$(A-Epprr)q+p\sqrt{A(A+Crr+Er^4)}+r\sqrt{A(A+Cpp+Ep^4)}=0$$

seu

$$(A-Eppqq)r+q\sqrt{A(A+Cpp+Ep^4)}+p\sqrt{A(A+Cqq+Eq^4)}=0$$

seu

$$(A-Eqqrr)p+r\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Crr+Er^4)}=0$$

quae oriuntur ex hac aequatione

$$A(pp+qq-rr)-Eppqqrr+2pq\sqrt{A(A+Crr+Er^4)}=0.$$

Haec vero ad rationalitatem perducta fit

$$AA(p^4+q^4+r^4-2ppqq-2pprr-2qqrr)$$
$$-2AEppqqrr(pp+qq+rr)-4ACppqqrr$$
$$+EEp^4q^4r^4=0,$$

quae autem ob pluralitatem radicum satisfacit omnibus signorum variationibus in superiori aequatione transcendente.

Corollarium 1.

613. Sumamus r negatiue, vt fiat

$$\Pi:r=\Pi:p+\Pi:q,$$

eritque

$$r=\frac{p\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Cpp+Ep^4)}}{A-Eppqq};$$

vnde colligitur fore

$$\sqrt{\frac{A+Crr+Er^4}{A}}=\frac{(A+Eppqq)\sqrt{(A+Cpp+Ep^4)(A+Cqq+Eq^4)}+2AEpq(pp+qq)+Cpq(A+Eppqq)}{(A-Eppqq)^2}.$$

Corollarium 2.

614. Quodsi ergo ponamus $q=p$, vt sit

$$\Pi:r=2\,\Pi:p,$$

erit

$$r=\frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4},$$

atque

$$\sqrt{\frac{A+Crr+Er^4}{A}}=\frac{AA+2ACpp+6AEp^4+2CEp^6+EEp^8}{(A-Ep^4)^2}.$$

Hoc igitur modo functio assignari potest aequalis duplo similis functionis.

Corollarium 3.

615. Si ponatur $q=\frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4}$ et

$$\sqrt{A(A+Cqq+Eq^4)}=\frac{A(AA+2ACpp+6AEp^4+2CEp^6+EEp^8)}{(A-Ep^4)^2},$$

vt sit $\Pi:q=2\,\Pi:p$, fiet ex primo Coroll. $\Pi:r=3\,\Pi:p$.

Tum

Tum igitur erit

$$r = \frac{p(3AA + 4ACpp + 6AEp^4 - EEp^8)}{AA - 6AEp^4 - 4CEp^6 - 3EEp^8}.$$

Scholion 1.

616. Nimis operofum eft hanc functionum multiplicationem vlterius continuare, multoque minus legem in earum progreffione deprehendere licet. Quodfi ponamus breuitatis gratia

$$\sqrt{A(A + Cpp + Ep^4)} = AP \text{ et } A - Ep^4 = A\mathfrak{P},$$

vt fit

$$Cpp = APP - A - Ep^4 \text{ et } Ep^4 = A(1 - \mathfrak{P}),$$

hac multiplicationes vsque ad quadruplum ita fe habebunt; fcilicet fi ftatuamus

$$\Pi : r = 2\,\Pi : p;\ \Pi : s = 3\,\Pi : p \text{ et } \Pi : t = 4\,\Pi : p$$

reperietur:

$$r = \frac{2Pp}{\mathfrak{P}},\ s = \frac{p(4PP - \mathfrak{P}\mathfrak{P})}{\mathfrak{P}\mathfrak{P} - 4PP(1 - \mathfrak{P})},\ t = \frac{4pP\mathfrak{P}[2PP(2 - \mathfrak{P}) - \mathfrak{P}\mathfrak{P}]}{\mathfrak{P}^4 - 16P^4(1 - \mathfrak{P})}.$$

Quodfi fimili modo ponamus

$$\sqrt{A(A + Crr + Er^4)} = AR \text{ et } A - Er^4 = A\mathfrak{R},$$

erit

$$R = \frac{2PP(2 - \mathfrak{P}) - \mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}} \text{ et } \mathfrak{R} = \frac{\mathfrak{P}^4 - 16P^4(1 - \mathfrak{P})}{\mathfrak{P}^4};$$

vnde pro quadruplicatione fit

$$t = \frac{2Rr}{\mathfrak{R}},\ T = \frac{2RR(2 - \mathfrak{R}) - \mathfrak{R}\mathfrak{R}}{\mathfrak{R}\mathfrak{R}},\ \mathfrak{T} = \frac{\mathfrak{R}^4 - 16R^4(1 - \mathfrak{R})}{\mathfrak{R}^4}.$$

Quare fi pro octuplicatione ftatuamus $\Pi : z = 8\,\Pi : p$ erit

$$z = \frac{2Tt}{\mathfrak{T}} = \frac{4rR\mathfrak{R}[2RR(2 - \mathfrak{R}) - \mathfrak{R}\mathfrak{R}]}{\mathfrak{R}^4 - 16R^4(1 - \mathfrak{R})}.$$

Hinc intelligitur quomodo in continua duplicatione verfari oporteat, neque tamen legem progreffionis obferuare licet. Caeterum cognitio huius legis ad incrementum Analyfeos maxime effet optanda, vt inde generatim relatio inter z et p, pro

aequalitate $\Pi : z = n\,\Pi : p$ definiri posset, quemadmodum hoc in capite praecedente successit; hinc enim eximias proprietates circa integralia formae $\int \frac{\partial z}{\sqrt{(A + Czz + Ez^4)}}$ cognoscere liceret, quibus scientia analytica haud mediocriter promoueretur.

Scholion 2.

617. Modus maxime idoneus in legem progressionis inquirendi, videtur, si ternos terminos se ordine excipientes contemplemur hoc modo

$$\Pi : x = (n-1)\Pi : p,\quad \Pi : y = n\,\Pi : p,\quad \Pi : z = (n+1)\Pi : p;$$

vbi cum sit

$$\Pi : x = \Pi : y - \Pi : p \text{ et } \Pi : z = \Pi : y + \Pi : p, \text{ erit}$$

$$x = \frac{y\sqrt{A(A+Cpp+Ep^4)} - p\sqrt{A(A+Cyy+Ey^4)}}{A - Eppyy}$$

$$z = \frac{y\sqrt{A(A+Cpp+Ep^4)} + p\sqrt{A(A+Cyy+Ey^4)}}{A - Eppyy};$$

vnde concludimus

$$(A - Eppyy)(x+z) = 2y\sqrt{A(A+Cpp+Ep^4)}.$$

Ponamus vt ante

$$\sqrt{A(A+Cpp+Ep^4)} = A\,P \text{ et } A - Ep^4 = A\,\mathfrak{P},$$

et quia singulae quantitates x, y, z factorem p simpliciter inuoluunt, sit

$$x = p\,X,\ y = p\,Y \text{ et } z = p\,Z;$$

erit

$$[1 - (1 - \mathfrak{P})\,Y\,Y](X + Z) = 2\,P\,Y$$

seu

$$Z = \frac{2PY}{1 - (1-\mathfrak{P})YY} - X,$$

cuius formulae ope ex binis terminis contiguis X et Y sequens Z haud difficulter inuenitur. Quod quo facilius appareat, ponatur $2P = Q$ et $1 - \mathfrak{P} = \Omega$, vt sit $Z = \frac{QY}{1-\Omega YY} - X$. Iam progressio quaesita ita se habebit

1)

1) 1;

2) $\frac{Q}{\mathfrak{P}}$;

3) $\frac{QQ-\mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}-QQ\mathfrak{Q}}$;

4) $\frac{Q^3\mathfrak{P}(1+\mathfrak{Q})-2Q\mathfrak{P}^3}{\mathfrak{P}^4-Q^4\mathfrak{Q}}$;

5) $\frac{\mathfrak{P}^6-3QQ\mathfrak{P}^4+Q^4\mathfrak{P}^2(1+2\mathfrak{Q})-Q^6\mathfrak{Q}\mathfrak{Q}}{\mathfrak{P}^6-3QQ\mathfrak{P}^4\mathfrak{Q}+Q^4\mathfrak{P}\mathfrak{P}\mathfrak{Q}(2+\mathfrak{Q})-Q^6\mathfrak{Q}}$ etc.

Quaeſtio ergo huc redit, vt inueſtigetur progreſſio, ex data relatione inter ternos terminos ſucceſſiuos X, Y, Z, quae ſit $Z=\frac{QY}{1-\mathfrak{Q}YY}-X$; exiſtente termino primo $=1$ et ſecundo $=\frac{Q}{1-\mathfrak{Q}}$.

Problema 80.

618. Si $\Pi:z$ eiusmodi denotet functionem ipſius z, vt ſit $\Pi:z=\int\frac{\partial z(L+Mzz+Nz^4)}{\sqrt{(A+Czz+Ez^4)}}$, integrali ita ſumto vt euaneſcat poſito $z=0$, comparationem inter huiusmodi functiones tranſcendentes inueſtigare.

Solutio.

Stabilita inter binas variabiles x et y hac relatione, vt ſit

$$Ay+\mathfrak{B}x-Ebbxxy=b\sqrt{A(A+Cxx+Ex^4)}\text{ ſeu}$$

$$Ax+\mathfrak{B}y-Ebbxyy=b\sqrt{A(A+Cyy+Ey^4)},$$

ſiue ſublata irrationalitate

$$A(xx+yy-bb)+2\mathfrak{B}xy-Ebbxxyy=0,$$

exiſtente breuitatis gratia $\mathfrak{B}=\sqrt{A(A+Cbb+Eb^4)}$, erit vti ante vidimus

$$\frac{\partial x}{\sqrt{(A+Cxx+Ex^4)}}+\frac{\partial y}{\sqrt{(A+Cyy+Ey^4)}}=0.$$

Ponamus igitur

$$\frac{\partial x(L+Mxx+Nx^4)}{\sqrt{(A+Cxx+Ex^4)}}+\frac{\partial y(L+Myy+Ny^4)}{\sqrt{(A+Cyy+Ey^4)}}=b\,\partial V\sqrt{A},$$

 vt

vt ſit noſtro ſignandi more

$$\Pi : x + \Pi : y = \text{Conſt.} + b\, V \sqrt{A},$$

vbi conſtans ita definiri debet, vt poſito $x = 0$ fiat $y = b$. Quaeſtio ergo ad inuentionem functionis V reuocatur; quem in finem loco ∂y valore ex priori aequatione ſubſtituto, erit

$$b\, \partial V \sqrt{A} = \frac{\partial x\, [M(xx - yy) + N(x^4 - y^4)]}{\sqrt{(A + Cxx + Ex^4)}};$$

verum quia

$$b \sqrt{A(A + Cxx + Ex^4)} = Ay + \mathfrak{B}x - Ebbxxy,$$

habebimus

$$\partial V = \frac{\partial x\, (xx - yy)\, [M + N(xx + yy)]}{Ay + \mathfrak{B}x - Ebbxxy}.$$

Sumamus iam aequationem rationalem

$$A(xx + yy - bb) + 2\mathfrak{B}xy - Ebbxxyy = 0,$$

et ponamus

$$xx + yy = tt \text{ et } xy = u,$$

vt ſit

$$A(tt - bb) + 2\mathfrak{B}u - Ebbuu = 0,$$

ideoque

$$At\, \partial t = -\mathfrak{B}\, \partial u + Ebbu\, \partial u.$$

Cum porro ſit

$$x\, \partial x + y\, \partial y = t\, \partial t \text{ et } x\, \partial y + y\, \partial x = \partial u,$$

erit

$$(xx - yy)\, \partial x = xt\, \partial t - y\, \partial u$$

ſeu

$$A(xx - yy)\, \partial x = -\partial u\, (Ay + \mathfrak{B}x - Ebbxxy),$$

ita vt ſit

$$\frac{\partial x\, (xx - yy)}{Ay + \mathfrak{B}x - Ebbxxy} = -\frac{\partial u}{A},$$

ex

ex quo deducitur

$$\partial V = -\frac{\partial u}{A}(M + Ntt),$$

et ob

$$tt = bb - \frac{2\mathfrak{B}u}{A} + \frac{Ebbuu}{A}, \text{ erit}$$

$$\partial V = -\frac{\partial u}{AA}(AM + ANbb - 2\mathfrak{B}Nu + ENbbuu):$$

vnde integrando elicitur

$$V = -\frac{Mu}{A} - \frac{Nbbu}{A} + \frac{\mathfrak{B}Nuu}{AA} - \frac{ENbbu^3}{3AA}.$$

Hoc ergo valore ſubſtituto, ob $u = xy$, habebimus

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} - \frac{Nb^3xy}{\sqrt{A}} + \frac{\mathfrak{B}Nbx^2y^2}{A\sqrt{A}} - \frac{ENb^3x^3y^3}{3A\sqrt{A}}.$$

Cum autem ſit

$$\mathfrak{B}xy = \tfrac{1}{2}Abb - \tfrac{1}{2}A(xx + yy) + Ebbxxyy$$

erit

$$\Pi x + \Pi y = \Pi b - \frac{Mbxy}{\sqrt{A}} - \frac{Nbxy}{2A\sqrt{A}}[A(bb + xx + yy) - \tfrac{1}{3}Ebbxxyy]$$

cui ergo aequationi ſatisfit per formulas algebraicas ſupra exhibitas, quibus relatio inter x, y et b exprimitur. Quodſi ergo ſtatuatur haec aequatio

$$\Pi : p + \Pi : q + \Pi : r$$
$$= \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}}[A(pp + qq + rr) - \tfrac{1}{3}Eppqqrr]$$

ea efficitur ſequenti relatione inter p, q, r conſtita

$$(A - Eppqq)r + p\sqrt{A(A + Cqq + Eq^4)} + q\sqrt{A(A + Cpp + Ep^4)} = 0$$

ſeu

$$(A - Epprr)q + p\sqrt{A(A + Crr + Er^4)} + r\sqrt{A(A + Cpp + Ep^4)} = 0$$

ſeu

$$(A - Eqqrr)p + q\sqrt{A(A + Crr + Er^4)} + r\sqrt{A(A + Cqq + Eq^4)} = 0$$

ſiue per ſimplicem irrationalitatem

$$A(pp + qq + rr) + 2pq\sqrt{A(A + Crr + Er^4)} - Eppqqrr = 0$$

ſeu

seu

$$A(pp+rr-qq)+2pr\sqrt{A(A+Cqq+Eq^4)}-Eppqqrr=0$$

seu

$$A(qq+rr-pp)+2qr\sqrt{A(A+Cpp+Ep^4)}-Eppqqrr=0$$

penitusque irrationalitate sublata

$$EEp^4q^4r^4-2AEppqqrr(pp+qq+rr)-4ACppqqrr$$
$$+AA(p^4+q^4+r^4-2ppqq-2pprr-2qqrr)=0.$$

Corollarium 1.

619. Sit $q=r=s$, vt habeamus hanc aequationem

$$\Pi:p+2\Pi:s=\frac{Mpss}{\sqrt{A}}+\frac{Npss}{2A\sqrt{A}}[A(pp+2ss)-\tfrac{1}{3}Epps^4]$$

cui satisfacit haec relatio

$$(A-Es^4)p+2s\sqrt{A(A+Css+Es^4)}=0.$$

Corollarium 2.

620. Sumamus s negatiue, et loco p substituamus ibi hunc valorem, vt habeamus

$$2\Pi:s+\Pi:q+\Pi:r+\frac{Mpss}{\sqrt{A}}+\frac{Npss}{2A\sqrt{A}}[A(pp+2ss)-\tfrac{1}{3}Epps^4]$$
$$=\frac{Mpqr}{\sqrt{A}}+\frac{Npqr}{2A\sqrt{A}}[A(pp+qq+rr)-\tfrac{1}{3}Eppqqrr)$$

existente

$$p=\frac{2s\sqrt{A(A+Css+Es^4)}}{A-Es^4},$$

vnde fit

$$\sqrt{A(A+Cpp+Ep^4)}=\frac{A(A+Css+Es^4)^2+A(4AE-CC)s^4}{(AE-s^4)^2}$$

qui valores in superioribus formulis substitui debent.

Corollarium 3.

621. Hoc modo effici poterit, vt partes algebraicae euanescant, atque functiones transcendentes solae inter se comparen-

parentur. Veluti si esset $N = 0$, statui oporteret $ss = qr$, vt fieret

$$2\Pi : s + \Pi : q + \Pi : r = 0.$$

At posito $ss = qr$, fit

$$p = \frac{2\sqrt{Aqr(A + Cqr + Eqqrr)}}{A - Eqqrr}.$$

Est vero etiam

$$p = \frac{-q\sqrt{A(A + Crr + Er^4)} - r\sqrt{A(A + Cqq + Eq^4)}}{A - Eqqrr},$$

quibus valoribus aequatis, oritur haec aequatio

$$(AA + EEq^4r^4)(qq - 6qr + rr) - 8Cqqrr(A + Eqqrr) \\ - 2AEqqrr(qq + 10qr + rr) = 0.$$

Scholion.

622. Si $\Pi : z$ exprimat arcum cuiuspiam lineae curvae respondentem abscissae vel cordae z, hinc plures arcus eiusdem curuae inter se comparare licet, vt vel differentia binorum arcuum fiat algebraica, vel arcus exhibeantur datam rationem inter se tenentes. Hoc modo eiusmodi insignes curvarum proprietates eruuntur, quarum ratio aliunde vix perspici queat. Comparatio quidem arcuum circularium ex elementis nota per caput praecedens, vt vidimus, facile expeditur, vnde etiam comparatio arcuum parabolicorum deriuatur. Ex hoc autem capite comparatio arcuum ellipticorum et hyperbolicorum simili modo institui potest; cum enim in genere arcus sectionis conicae tali formula exprimatur $\int \partial x \sqrt{\frac{a + bxx}{c + exx}}$, haec transformata in istam $\int \frac{\partial x(a + bxx)}{\sqrt{(ac + (ae + bc)xx + bex^4)}}$, per praecepta tradita tractari potest, ponendo $A = ac$, $C = ab + bc$, et $E = be$, $L = a$, $M = b$ atque $N = 0$. Haec autem investigatio ad formulas, quarum denominator est

$$\sqrt{(A + 2Bz + Czz + Dz^3 + Ez^4)}$$

extendi poteſt, ſimilisque eſt praecedenti, quam idcirco hic ſum expoſiturus, vnde ſimul patebit, hunc eſſe vltimum terminum, quousque progredi liceat. Formulae enim integrales magis complicatae, vbi poſt ſignum radicale altiores poteſtates ipſius z occurrunt, vel ipſum ſignum radicale altiorem dignitatem inuoluit, hoc modo non videntur inter ſe comparari poſſe, pauciſſimis caſibus exceptis, qui per quampiam ſubſtitutionem ad huiusmodi formam reduci queant.

Problema 81.

623. Si $\Pi:z$ eiusmodi functionem ipſius z denotet, vt ſit

$$\Pi:z = \frac{\partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}},$$

huiusmodi functiones inter ſe comparare.

Solutio.

Inter binas variabiles x et y ſtatuatur relatio hac aequatione expreſſa

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy = 0,$$

vnde cum fiat

$$yy = \frac{-2y(\beta + \delta x + \varepsilon xx) - \alpha - 2\beta x - \gamma xx}{\gamma + 2\varepsilon x + \zeta xx},$$

erit radice extracta

$$y = \frac{-\beta - \delta x - \varepsilon xx + \sqrt{[(\beta + \delta x + \varepsilon xx)^2 - (\alpha + 2\beta x + \gamma xx)(\gamma + 2\varepsilon x + \zeta xx)]}}{\gamma + 2\varepsilon x + \zeta xx}.$$

Reducatur ſignum radicale ad formam propoſitam, ponendo

$$\beta\beta - \alpha\gamma = Am,\ \beta\delta - \alpha\varepsilon - \beta\gamma = Bm,$$
$$\delta\delta - 2\beta\varepsilon - \alpha\zeta - \gamma\gamma = Cm,\ \delta\varepsilon - \beta\zeta - \gamma\varepsilon = Dm,$$
$$\varepsilon\varepsilon - \gamma\zeta = Em;$$

vnde ex ſex coefficientibus α, β, γ, δ, ε, ζ, quinque definiuntur, atque ad ſextum inſuper accedit littera m, ita vt aequatio

quatio assumta adhuc constantem arbitrariam inuoluat. Inde ergo si breuitatis gratia ponamus

$$\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)} = X \text{ et}$$
$$\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)} = Y,$$

habebimus

$$\beta + \gamma y + \delta x + \varepsilon xx + 2\varepsilon xy + \zeta xxy = X\sqrt{m} \text{ et}$$
$$\beta + \gamma x + \delta y + \varepsilon yy + 2\varepsilon xy + \zeta xyy = Y\sqrt{m}.$$

At aequatio assumta per differentiationem dat

$$+\partial x(\beta + \gamma x + \delta y + 2\varepsilon xy + \varepsilon yy + \zeta xyy)$$
$$+\partial y(\beta + \gamma y + \delta x + \varepsilon xx + 2\varepsilon xy + \zeta xxy) = 0,$$

quae expressiones quia cum superioribus conueniunt, dant,

$$Y\partial x\sqrt{m} + X\partial y\sqrt{m} = 0, \text{ seu } \frac{\partial x}{X} + \frac{\partial y}{Y} = 0:$$

vnde integrando colligimus

$$\Pi : x + \Pi : y = \text{Const.}$$

quae constans, si posito $x = 0$ fiat $y = b$, erit $= \Pi : 0 + \Pi : b$; vel in genere, si posito $x = a$ fiat $y = b$, ea erit $= \Pi : a + \Pi : b$. Quodsi ergo litterae α, β, γ, δ, ε, ζ per conditiones superiores definiantur, aequatio assumta algebraica inter x et y erit integrale completum huius aequationis differentialis

$$\frac{\partial x}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} + \frac{\partial y}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = 0.$$

Corollarium 1.

624. Ad has litteras α, β, γ, δ, ε, ζ definiendas, sumantur primo aequationes binae ad dextram positae, quae sunt

$$(\delta - \gamma)\beta - \alpha\varepsilon = Bm \text{ et } (\delta - \gamma)\varepsilon - \zeta\beta = Dm,$$

vnde quaerantur binae β et ε, reperieturque

$$\beta = \frac{(\delta - \gamma)B + \alpha D}{(\delta - \gamma)^2 - \alpha\zeta} m \text{ et } \frac{(\delta - \gamma)D + \zeta B}{(\delta - \gamma)^2 - \alpha\zeta} m.$$

Corollarium 2.

625. Sit breuitatis gratia $\delta - \gamma = \lambda$ seu $\delta = \gamma + \lambda$, erit

$$\beta = \frac{D\alpha + B\lambda}{\lambda\lambda - \alpha\zeta} m \text{ et } \varepsilon = \frac{B\zeta + D\lambda}{\lambda\lambda - \alpha\zeta} m.$$

Iam ex conditione prima et vltima oritur

$$\beta\beta\zeta - \alpha\varepsilon\varepsilon = (A\zeta - E\alpha) m,$$

vbi illi valores substituti praebent

$$\frac{BB\zeta - DD\alpha}{\lambda\lambda - \alpha\zeta} m = A\zeta - E\alpha,$$

vnde fit

$$m = \frac{(\lambda\lambda - \alpha\zeta)(A\zeta - E\alpha)}{BB\zeta - DD\alpha}.$$

At ex prima et vltima sequitur

$$DD\beta\beta - BB\varepsilon\varepsilon + \gamma(BB\zeta - DD\alpha) = (ADD - BBE) m$$

vnde colligitur

$$\gamma = \frac{[(A\zeta - E\alpha)(ADD - BBE)\lambda\lambda + 2BD(A\zeta - E\alpha)\lambda + ABB\zeta\zeta - DDE\alpha\alpha}{(BB\zeta - DD\alpha)^2}.$$

Corollarium 3.

626. Superest tertia aequatio

$$2\gamma\lambda + \lambda\lambda - 2\beta\varepsilon - \alpha\zeta = C m$$

quae, cum pro m substituto valore fit

$$\beta = \frac{(A\zeta - E\alpha)(D\alpha + B\lambda)}{BB\zeta - DD\alpha} \text{ et } \varepsilon = \frac{(A\zeta - E\alpha)(B\zeta + D\lambda)}{BB\zeta - DD\alpha},$$

si isti valores substituantur, commode inde colligitur

$$\lambda = \frac{C(A\zeta - E\alpha)(BB\zeta - DD\alpha) - 2BD(A\zeta - E\alpha)^2 - (BB\zeta - DD\alpha)^2}{2(A\zeta - A\alpha)(ADD - BBE)}.$$

Scholion.

627. Quia his valoribus vti non licet, quoties fuerit $ADD - BBE = 0$, aliam resolutionem huic incommodo non obnoxiam tradam. Posito $\delta = \gamma + \lambda$, fit insuper $\lambda\lambda = \alpha\zeta + \mu$, vt primae formulae fiant

$\beta =$

$$\beta = \frac{m}{\mu}(D\alpha + B\lambda) \text{ et } \varepsilon = \frac{m}{\mu}(B\zeta + D\lambda).$$

Iam prima et vltima iunctis prodit

$$A\zeta - E\alpha = \frac{m}{\mu}(BB\zeta - DD\alpha)$$

qua aequatione ratio inter α et ζ definitur, quae cum sufficiat, erit

$$\alpha = \mu A - BBm \text{ et } \zeta = \mu E - DDm,$$

hincque

$$\lambda\lambda = \mu + (\mu A - BBm)(\mu E - DDm):$$

vnde colligimus

$$\gamma = \frac{mm}{\mu\mu}[2BD\lambda + (ADD - BBE)\mu] - \frac{2BBDDm^3}{\mu\mu} - \frac{m}{\mu}.$$

Valores α et ζ in formula Corollarii 3. substituti dant

$$\lambda = \frac{\mu\mu}{2m} + BDm - \tfrac{1}{2}C\mu,$$

cuius quadratum illi valori $\alpha\zeta + \mu$ aequatum, perducit ad hanc aequationem

$$\mu(\mu - Cm)^2 + 4(BD - AE)mm\mu$$
$$+ 4(ADD - BCD + BCE)m^3 = 4mm,$$

ad quam resoluendam ponatur $\mu = Mm$, fietque

$$m = \frac{4}{M(M-C)^2 + 4M(BD - AE) + 4(ADD - BCD + BBE)},$$

atque hic est M constans illa arbitraria pro integrali completo requisita. Hoc modo omnes litterae α, β, γ, δ, ε, ζ eodem denominatore affecti prodibunt, quo omisso habebimus

$$\alpha = 4(AM - BB),\ \beta = 2B(M - C) + 4AD,\ \gamma = 4AE - (M - C)^2,$$
$$\zeta = 4(EM - DD),\ \varepsilon = 2D(M - C) + 4BE,$$
$$\delta = MM - CC + 4(AE + BD),$$

quibus inuentis aequatio nostra canonica

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy$$
$$+ 2\varepsilon xy(x + y) + \zeta xxyy$$

si breuitatis gratia ponamus

$$M(M-C)^2+4M(BD-AE)+4(ADD-BCD+BBE)=\Delta,$$

resoluta dabit

$$\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)=$$
$$\pm 2\sqrt{\Delta}(A+2Bx+Cxx+2Dx^3+Ex^4)$$
$$\beta+\delta y+\varepsilon yy+x(\gamma+2\varepsilon y+\zeta yy)=$$
$$\pm 2\sqrt{\Delta}(A+2By+Cyy+2Dy^3+Ey^4),$$

quae ergo est integrale completum huius aequationis differentialis

$$0=\frac{\partial x}{\pm\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}+\frac{\partial y}{\pm\sqrt{(A+2By+Cyy+2Dz^3+Ey^4)}}.$$

Scholion.

628. Cum hic ab idonea coefficientium determinatione totum negotium pendeat, operae praetium erit, eam luculentius exponere. Posito igitur statim

$$\delta=\gamma+\lambda \text{ et } \lambda\lambda-\alpha\zeta=Mm,$$

quinque conditiones adimplendae sunt:

I. $\beta\beta-\alpha\gamma=Am;$

II. $\varepsilon\varepsilon-\gamma\zeta=Em;$

III. $\beta\lambda-\alpha\varepsilon=Bm;$

IV. $\varepsilon\lambda-\beta\zeta=Dm;$

V. $Mm+2\gamma\lambda-2\beta\varepsilon=Cm.$

Hinc ex tertia et quarta combinando deducitur

$$m(B\lambda+D\alpha)=\beta(\lambda\lambda-\alpha\zeta)=\beta Mm, \text{ ergo } \beta=\frac{B\lambda+D\alpha}{M},$$
$$m(D\lambda+B\zeta)=\varepsilon(\lambda\lambda-\alpha\zeta)=\varepsilon Mm \text{ ergo } \varepsilon=\frac{D\lambda+B\zeta}{M}.$$

Iam ex prima et secunda elidendo γ, oritur

$$m(A\zeta-E\alpha)=\beta\beta\zeta-\varepsilon\varepsilon\alpha=\frac{BB\zeta-DD\alpha}{M}.m$$

hinc-

hincque

$$\zeta(AM - BB) = \alpha(EM - DD);$$

quare statuatur

$$\alpha = n(AM - BB) \text{ et } \zeta = n(EM - DD).$$

Tum vero indidem est

$$E\beta\beta - E\alpha\gamma = A\varepsilon\varepsilon - A\gamma\zeta, \text{ seu}$$

$$\gamma(A\zeta - E\alpha) = A\varepsilon\varepsilon - E\beta\beta:$$

pro qua tractanda cum sit, pro α et ζ substitutis valoribus,

$$\beta = nAD + \tfrac{B}{M}(\lambda - nBD) \text{ et } \varepsilon = nBE + \tfrac{D}{M}(\lambda - nBD),$$

sit breuitatis ergo $\lambda - nBD = nMN$, vt habeamus

$$\beta = n(AD + BN) \text{ et } \varepsilon = n(BE + DN),$$

et quia

$$A\zeta - E\alpha = n(BBE - ADD)$$

atque

$$A\varepsilon\varepsilon - E\beta\beta = nn(ABBEE + ADDNN - AADDE - BBENN), \text{ seu}$$

$$A\varepsilon\varepsilon - E\beta\beta = nn(BBE - ADD)(AE - NN) \text{ fiet,}$$

$$\gamma = n(AE - NN).$$

Cum autem sit

$$\lambda = n(BD + MN) \text{ et}$$

$$\lambda\lambda = nn(AM - BB)(EM - DD) + Mm, \text{ erit}$$

$$Mm = nn[2BDMN + MMNN - AEMM + M(ADD + BBE)]$$

seu

$$m = nn(2BDN + MNN - AEM + ADD + BBE).$$

Denique aequatio quinta $\beta\varepsilon - \gamma\lambda = \frac{1}{2}m(M - C)$ euoluta praebet

$$\beta\varepsilon - \gamma\lambda = nn[(AD + BN)(BE + DN) - (AE - NN)(BD + MN)]$$
$$- nnN(2BDN + MNN - AEM + ADD + BBE) = Nm,$$

vnde

vnde fit $N = \frac{1}{2}(M - C)$, ac propterea

$$m = nn[BD(M-C) + \tfrac{1}{4}M(M-C)^2 - AEM + ADD + BBE].$$

Hincque sumendo $n = 4$ superiores valores obtinentur.

Exemplum 1.

629. *Inuenire integrale completum huius aequationis differentialis*

$$\frac{\partial p}{\pm\sqrt{(a+bp)}} + \frac{\partial q}{\pm\sqrt{(a+bq)}} = 0.$$

Hic est $x = p$, $y = q$, $A = a$, $B = \frac{1}{2}b$, $C = 0$, $D = 0$, $E = 0$; vnde fiunt coefficientes

$$\alpha = 4aM - bb,\ \beta = bM,\ \gamma = -MM,$$
$$\zeta = 0,\qquad \varepsilon = 0,\quad \delta = MM,$$

et $\Delta = M^3$, vnde integrale completum erit

$$bM + MMp - MMq = \pm 2M\sqrt{M(a+bp)},\ \text{seu}$$
$$b + M(p-q) = \pm 2\sqrt{M(a+bp)},\ \text{vel}$$
$$b + M(q-p) = \pm 2\sqrt{M(a+bq)};$$

quae signa ambigua radicalium cum signis in aequatione differentiali conuenire debent.

Exemplum 2.

630. *Inuenire integrale completum huius aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a+bp^2)}} + \frac{\partial p}{\pm\sqrt{(a+bq^2)}} = 0.$

Sumto $x = p$ et $y = q$, erit $A = a$, $B = 0$, $C = b$, $D = 0$, ergo

$$\alpha = 4aM,\ \beta = 0,\ \gamma = -(M-b)^2,$$
$$\zeta = 0,\quad \varepsilon = 0,\ \delta = MM - bb,$$

atque $\Delta = M(M-b)^2$;

vnde integrale completum in his aequationibus continebitur:

(MM

$(MM-bb)p-(M-b)^2 q = \pm 2(M-b)\sqrt{M(a+bpp)}$, seu
$(M+b)p-(M-b)q = \pm 2\sqrt{M(a+bpp)}$ et
$(M+b)q-(M-b)p = \pm 2\sqrt{M(a+bqq)}$.

Exemplum 3.

631. *Inuenire integrale completum huius aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a+bp^3)}} + \frac{\partial q}{\pm\sqrt{(a+bq^3)}} = 0$.

Sumto $x = p$, $y = q$, erit $A = a$, $B = 0$, $C = 0$, $D = \frac{1}{2}b$, $E = 0$, ergo

$\alpha = 4aM$, $\beta = 2ab$, $\gamma = -MM$,
$\zeta = -bb$, $\varepsilon = bM$, $\delta = MM$ et
$\Delta = M^3 + abb$;

vnde integrale completum

$$2ab + MMp + bMpp + q(-MM + 2bMp - bbpp) = \pm 2\sqrt{(M^3+abb)(a+bp^3)}$$

siue

$$2ab + Mp(M+bp) - q(M-bp)^2 = \pm 2\sqrt{(M^3+abb)(a+bp^3)}$$

et

$$2ab + Mq(M+bq) - p(M-bq)^2 = \pm 2\sqrt{(M^3+abb)(a+bq^3)}.$$

Exemplum 4.

632. *Inuenire integrale completum huius aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a+bp^4)}} + \frac{\partial q}{\pm\sqrt{(a+bq^4)}} = 0$.

Posito $x = p$, $y = q$, erit $A = a$, $B = 0$, $C = 0$, $D = 0$, $E = b$, ergo

$\alpha = 4aM$, $\beta = 0$, $\gamma = 4ab - MM$,
$\zeta = 4bM$, $\varepsilon = 0$, $\delta = MM + 4ab$, et
$\Delta = M^3 - 4abM$;

 vndo

vnde integrale completum

$$(MM + 4ab)p + q(4ab - MM + 4bMpp) = \pm 2\sqrt{M(MM - 4ab)(a + bp^4)}$$

$$(MM + 4ab)q + p(4ab - MM + 4bMqq) = \pm 2\sqrt{M(MM - 4ab)(a + bq^4)}.$$

Exemplum 5.

633. *Inuenire integrale completum huius aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a + bp^6)}} + \frac{\partial q}{\pm\sqrt{(a + bq^6)}} = 0.$

Ponatur $x = pp$ et $y = qq$, atque aequatio nostra generalis induet posito $A = 0$, hanc formam

$$\frac{\partial p}{\pm\sqrt{(2B + Cpp + 2Dp^4 + Ep^6)}} + \frac{\partial q}{\pm\sqrt{(2B + Cqq + 2Dq^4 + Eq^6)}} = 0.$$

Fieri ergo oportet $B = \frac{1}{2}a$, $C = 0$, $D = 0$ et $E = b$; vnde coefficientes ita determinantur

$$\alpha = -aa,\ \beta = aM,\ \gamma = -MM,$$
$$\zeta = 4bM,\ \varepsilon = 2ab,\ \delta = MM, \text{ et}$$
$$\Delta = M^3 + aab;$$

ergo integrale completum

$$aM + MMpp + 2abp^4 + qq(-MM + 4abpp + 4bMp^4) = \pm 2p\sqrt{(M^3 + aab)(a + bp^6)}$$

siue

$$aM + MMqq + 2abq^4 + pp(-MM + 4abqq + 4bMq^4) = \pm 2q\sqrt{(M^3 + aab)(a + bq^6)}.$$

Corollarium.

634. Si sumatur constans $M = -\sqrt[3]{aab}$, vt sit $M^3 + aab = 0$, prodibit integrale particulare, quod ita se habebit

pp

$$pp = \frac{qq\sqrt[3]{b}+\sqrt[3]{a}}{2qq\sqrt[3]{b}-\sqrt[3]{a}} : \sqrt[3]{\frac{a}{b}} \text{ seu } qq = \frac{pp\sqrt[3]{b}+\sqrt[3]{a}}{2pp\sqrt[3]{b}-\sqrt[3]{a}} : \sqrt[3]{\frac{a}{b}}$$

quod aequationi differentiali vtique satisfacit.

Problema 82.

635. Proposita hac aequatione differentiali

$$\frac{\partial p}{\pm\sqrt{(a+bpp+cp^4+ep^6)}} + \frac{\partial q}{\pm\sqrt{(a+bqq+cq^4+eq^6)}} = 0$$

eius integrale completum algebraice assignare.

Solutio.

Aequatio praecedens differentialis algebraice integrata ad hanc formam reducitur, ponendo $x = pp$ et $y = qq$, atque $A = 0$; prodibit enim

$$\frac{\partial p}{\pm\sqrt{(2B+Cpp+2Dp^4+Ep^6)}} + \frac{\partial q}{\pm\sqrt{(2B+Cqq+2Dq^4+Eq^6)}} = 0.$$

Quare tantum opus est vt fiat

$$A = 0,\ B = \tfrac{1}{2}a,\ C = b,\ D = \tfrac{1}{2}c,\ E = e,$$

vnde coefficientes $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ ita definientur

$$\alpha = -aa, \quad \beta = a(M-b), \quad \gamma = -(M-b)^2,$$
$$\zeta = 4eM-cc,\ \varepsilon = c(M-b)+2ae,\ \delta = MM-bb+ae,$$
$$\Delta = M(M-b)^2+acM-abc+aae =$$
$$(M-b)^3+b(M-b)^2+ac(M-b)+aae;$$

hincque integrale completum ob constantem M ab arbitrio nostro pendentem, erit,

$$\beta+\delta pp+\varepsilon p^4+qq(\gamma+2\varepsilon pp+\zeta p^4) =$$
$$\pm 2p\sqrt{\Delta(a+bp^2+cp^4+ep^6)}$$
$$\beta+\delta qq+\varepsilon q^4+pp(\gamma+2\varepsilon qq+\zeta q^4) =$$
$$\pm 2q\sqrt{\Delta(a+bq^2+cq^4+eq^6)},$$

quae binae quidem aequationes inter se conueniunt, sed ob am-

ambiguitatem ſignorum in ipſa aequatione differentiali ambae notari debent, ambiguitate inde ſublata. Vtrinque autem haec aequatio rationalis reſultat

$$c = \alpha + 2\beta(pp+qq) + \gamma(p^4+q^4) + 2\delta ppqq + 2\varepsilon ppqq(pp+qq) + \zeta p^4 q^4.$$

Corollarium 1.

636. Si conſtans M ita ſumatur, vt fiat $\Delta = 0$, obtinetur integrale particulare huius formae $qq = \frac{E+Fpp}{G+Hpp}$, quod etiam a poſteriori cognoſcere licet. Vt enim ſatisfaciat ſumi debet

$$aG^3 + bEGG + cEEG + eE^3 = 0,$$

vnde ratio $E:G$ definitur, tum vero inuenitur $F = -G$ et denique

$$H = \frac{-cEG - 2eEE}{aG} = \frac{2aGG + 2bEG + cEE}{aE}.$$

Corollarium 2.

637. Conſtans M ita mutetur, vt ſit $M - b = \frac{a}{ff}$, fietque

$$\alpha = -aa, \qquad \beta = \frac{aa}{ff}, \qquad \gamma = -\frac{aa}{f^4},$$
$$\zeta = 4be - cc + \frac{4ae}{ff}, \quad \varepsilon = \frac{ac}{ff} + 2ae, \quad \delta = \frac{aa}{f^4} + \frac{2ab}{ff} + ac, \text{ et}$$
$$\Delta = \frac{aa}{f^6}(a + bff + cf^4 + ef^6),$$

et aequatio integralis erit

$$aaff + a(a+2bff+cf^4)pp + aff(c+2eff)p^4$$
$$-qq[aa - 2aff(c+2eff)pp + ff(ccff - 4beff - 4ae)p^4]$$
$$= \pm 2afp\sqrt{(a+bff+cf^4+ef^6)(a+bpp+cp^4+ep^6)};$$

vnde patet poſito $p = 0$ fore $qq = ff$.

Corol-

Corollarium 3.

638. Haec aequatio facile in hanc formam transmutatur

$$aff(a+bpp+cp^4+ep^6)+app(a+bff+cf^4+ef^6)$$
$$-qq(a-cffpp)^2-aeffpp(ff-pp)^2+4effppqq(aff+app+bffpp)$$
$$=\pm 2fp\surd a(a+bff+cf^4+ef^6)a(a+bpp+cp^4+ep^6);$$

vnde ſtatim patet ſi ſit $e=0$, fore hanc aequationem, radicem extrahendo

$$f\surd a(a+bpp+cp^4)\pm p\surd a(a+bff+cf^4)=q(a-cffpp)$$

quae eſt integralis completa huius differentialis

$$\frac{\partial p}{\pm\surd(a+bpp+cp^4)}+\frac{\partial q}{\pm\surd(a+bqq+cq^4)}=0$$

prorſus vt ſupra iam inuenimus.

Corollarium 4.

639. Simili modo patet in genere, quando e non euaneſcit, integrale completum ita commodius exprimi poſſe

$$[f\surd a(a+bpp+cp^4+ep^6)\pm p\surd a(a+bff+cf^4+ef^6)]^2=$$
$$qq(a-cffpp)^2+aeffpp(ff-pp)^2-4effppqq(aff+app+bffpp),$$

quae ergo cum poſito $p=0$ fiat $q=f$, reſpondet huic functionum tranſcendentium relationi

$$\pm\Pi:p\pm\Pi:q=\pm\Pi:0\pm\Pi:f.$$

Scholion 1.

640. Genera igitur functionum tranſcendentium, quas hoc modo perinde atque arcus circulares inter ſe comparare licet, in his binis formulis integralibus continentur

$$\int\frac{\partial z}{\surd(A+2Bz+Czz+2Dz^3+Ez^4)} \text{ et } \int\frac{\partial z}{\surd(a+bzz+cz^4+ez^6)}$$

neque haec methodus ad alias formas magis complexas extendi poſſe videtur. Neque etiam poſterior in denominatore po-

testates impares ipsius z admittit: nisi forte simplex substitutio reductioni ad illam formam sufficiat. Facile autem patet huiusmodi formam

$$\int \frac{\partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4 + Fz^5 + Gz^6)}},$$

hac methodo tractari certe non posse; si enim coefficientes ita essent comparati, vt radicis extractio succederet, talis formula $\int \frac{\partial z}{a + bz + czz + ez^3}$ prodiret, cuius integratio, cum tam logarithmos quam arcus circulares inuoluat, fieri omnino nequit, vt plures huiusmodl functiones algebraice inter se comparentur. Caeterum prior formula latius patet quam posterior, cum haec ex illa nascatur posito $A = 0$, si zz loco z scribatur. De priori autem notari meretur, quod eandem formam seruet, etiamsi transformetur hac substitutione $z = \frac{\alpha + \beta y}{\gamma + \delta y}$; prodit enim

$$\int \frac{(\beta\gamma - \alpha\delta)\partial y}{\sqrt{[A(\gamma+\delta y)^4 + 2B(\alpha+\beta y)(\gamma+\delta y)^3 + C(\alpha+\beta y)^2(\gamma+\delta y)^2 + 2D(\alpha+\beta y)^3(\gamma+\delta y) + E(\alpha+\beta y)^4]}},$$

ex quo intelligitur quantitates α, β, γ, δ, ita accipi posse, vt potestates impares euanescant. Vel etiam ita definiri poterunt, vt terminus primus et vltimus euanescat, tum enim posito $y = uu$, iterum forma a potestatibus imparibus immunis nascitur.

Scholion 2.

641. Sublatio autem potestatum imparium ita commodissime instituitur. Cum formula

$$A + 2Bz + Czz + 2Dz^3 + Ez^4$$

certe semper habeat duos factores reales, ita exhibeatur formula integralis

$$\int \frac{\partial z}{\sqrt{(a + 2bz + czz)(f + 2gz + hzz)}},$$

quae posito $z = \frac{\alpha + \beta y}{\gamma + \delta y}$, abit in

$$\int \frac{(\beta\gamma - \alpha\delta)\partial y}{\sqrt{[a(\gamma+\delta y)^2 + 2b(\alpha+\beta y)(\gamma+\delta y) + c(\alpha+\beta y)^2][f(\gamma+\delta y)^2 + 2g(\alpha+\beta y)(\gamma+\delta y) + h(\alpha+\beta y)^2]}},$$

vbi

vbi denominatoris factores euoluti sunt

$$(a\gamma\gamma + 2b\alpha\gamma + c\alpha\alpha) + 2(a\gamma\delta + b\alpha\delta + b\beta\gamma + c\alpha\beta)y$$
$$+ (a\delta\delta + 2b\beta\delta + c\beta\beta)yy$$
$$(f\gamma\gamma + 2g\alpha\gamma + h\alpha\alpha) + 2(f\gamma\delta + g\alpha\delta + g\beta\gamma + h\alpha\beta)y$$
$$+ (f\delta\delta + 2g\beta\delta + h\beta\beta)yy$$

quodsi iam vtroque terminus medius euanescens reddatur, fit

$$\frac{\delta}{\beta} = \frac{-b\gamma - c\alpha}{a\gamma + b\alpha} = \frac{-g\gamma - h\alpha}{f\gamma + g\alpha},$$

hincque

$$bf\gamma\gamma + (bg + cf)\alpha\gamma + cg\alpha\alpha = ag\gamma\gamma + (ah + bg)\alpha\gamma + bh\alpha\alpha$$

seu

$$\gamma\gamma = \frac{(ah - cf)\alpha\gamma + (bh - cg)\alpha\alpha}{bf - ag},$$

vnde fit

$$\frac{\gamma}{\alpha} = \frac{ah - cf + \sqrt{[(ah - cf)^2 + 4(bf - ag)(bh - cg)]}}{2(bf - ag)}.$$

Hinc sufficere posset eas tantum formulas, in quibus potestates impares desunt, tractasse, id quod initio huius capitis fecimus, sed si insuper numerator accedat, haec reductio non amplius locum habet.

Problema 83.

642. Denotante n numerum integrum quemcunque, inuenire integrale completum algebraice expressum huius aequationis differentialis

$$\frac{\partial y}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = \frac{n\,\partial x}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}}.$$

Solutio.

Per functiones transcendentes integrale completum est

$$\Pi : y = n\,\Pi : x + \text{Const.}$$

At vt idem algebraice expressum eruamus, posito $M - C = L$, fit per formulas supra (627.) inuentas

$\alpha = 4$

$\alpha = 4(AC - BB + AL)$, $\beta = 4AD + 2BL$, $\gamma = 4AE - LL$,
$\zeta = 4(CE - DD + EL)$, $\varepsilon = 4BE + 2DL$, $\delta = 4AE + 4BD + 2CL + LL$,
et

$\Delta = L^3 + CL^2 + 4(BD - AE) + 4(ADD + BBE - ACE)$.

Quibus pofitis fi fuerit

$$\beta + \delta p + \varepsilon pp + q(\gamma + 2\varepsilon p + \zeta pp) = 2\surd\Delta(A + 2Bp + Cp^2 + 2Dp^3 + Ep^4)$$

$$\beta + \delta q + \varepsilon qq + p(\gamma + 2\varepsilon q + \zeta qq) = -2\surd\Delta(A + 2Bq + Cq^2 + 2Dq^3 + Eq^4)$$

erit $\Pi : q = \Pi : p +$ Conft.

Cum autem hae duae aequationes inter fe conueniant, et in hac rationali contineantur

$$\alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq = 0$$

fi fumamus, pofito $p = a$ fieri $q = b$, conftans illa L ita definiri debet, vt fit

$$\alpha + 2\beta(a+b) + \gamma(aa+bb) + 2\delta ab + 2\varepsilon ab(a+b) + \zeta aabb = 0,$$

eritque

$$\Pi : q = \Pi : p + \Pi : b - \Pi : a;$$

vbi iam nullum ineft difcrimen inter conftantes et variabiles. Ponamus ergo $p = b$, vt fit

$$\Pi : q = 2\Pi : p - \Pi : a$$

atque huic aequationi fuperiores aequationes algebraicae conveniunt, fi modo quantitas L ita definiatur, vt fit

$$\alpha + 2\beta(a+p) + \gamma(aa+pp) + 2\delta ap + 2\varepsilon ap(a+p) + \zeta aapp = 0,$$

vnde deducitur

$$\tfrac{1}{2} L (a-p)^2 = A + B (a+p) + C a p + D a p (a+p) + E a a p p$$
$$\pm \surd (A + 2Ba + Caa + 2Da^3 + Ea^4)(A + 2Bp + Cpp + 2Dp^3 + Ep^4).$$

Hoc ergo valore pro L conſtituto, indeque litteris α, β, γ, δ, ε, ζ per ſuperiores formulas rite definitis, ſi iam p et q vt variabiles, a vero vt conſtantem ſpectemus, erit haec aequatio

$$\alpha + 2\beta (p+q) + \gamma (pp+qq) + 2\delta pq + 2\varepsilon pq (p+q) + \zeta ppqq = 0,$$

integrale completum huius aequationis differentialis

$$\frac{\partial q}{\surd (A + 2Bq + Cqq + 2Dq^3 + Eq^4)} = \frac{2\partial p}{\surd (A + 2Bp + Cpp + 2Dp^3 + Ep^4)}.$$

Poſtquam hoc modo q per p definiuimus, determinetur r per hanc aequationem

$$\alpha + 2\beta (q+r) + \gamma (qq+rr) + 2\delta qr + 2\varepsilon qr (q+r) + \zeta qqrr = 0,$$

erit

$$\Pi : r - \Pi : q = \Pi : p - \Pi : a,$$

quoniam, poſito $q = a$ et $r = p$, littera L, quae in valores α, β, γ, δ, ε, ζ ingreditur, perinde definitur vt ante. Quare cum ſit

$$\Pi : q = 2\,\Pi : p - \Pi : a, \text{ erit } \Pi : r = 3\,\Pi : p - 2\,\Pi : a;$$

vnde ſumto a conſtante, illa aequatio algebraica inter q et r, dum q per praecedentem aequationem ex p definitur, erit integrale completum huius aequationis differentialis

$$\frac{\partial r}{\surd (A + 2Br + Crr + 2Dr^3 + Er^4)} = \frac{3\partial p}{\surd (A + 2Bp + Cpp + 2Dp^3 + Ep^4)}.$$

Hoc valore ipſius r per p inuento, quaeratur s per hanc aequationem

$$\alpha + 2\beta (r+s) + \gamma (rr+ss) + 2\delta rs + 2\varepsilon rs (r+s) + \zeta rrss = 0,$$

retinente L ſemper valorem primo aſſignatum, eritque

$$\Pi : s - \Pi : r = \Pi : p - \Pi : a, \text{ ſeu } \Pi : s = 4\,\Pi : p - 3\,\Pi : a$$

vnde iſta aequatio algebraica erit integrale completum huius

aequationis differentialis

$$\frac{\partial s}{\sqrt{(A+2Bs+Css+2Ds^3+Es^4)}} = \frac{4\partial p}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

Cum hoc modo quousque libuerit progredi liceat, perspicuum est, ad integrale completum huius aequationis differentialis inveniendum

$$\frac{\partial z}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}} = \frac{n\partial p}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

sequentes operationes institui oportere.

1.) Quaeratur quantitas L, vt sit

$$\tfrac{1}{2}L(p-a)^2 = A + B(a+p) + Cap + Dap(a+p) + Eaapp$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}$$

2.) Hinc determinentur litterae α, β, γ, δ, ε, ζ, per has formulas

$$\alpha = 4(AC-BB+AL),\ \beta = 4AD+2BL,\ \gamma = 4AE-LL,$$
$$\zeta = 4(CE-DD+EL),\ \varepsilon = 4BE+2DL,\ \delta = 4AE+4BD+2CL+LL.$$

3.) Formetur series quantitatum $p, q, r, s, t, \ldots\ldots z$, quarum prima sit p, secunda q, tertia r etc. vltima vero ordine n sit z, quae successiue per has aequationes determinentur

$$\alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq = 0$$
$$\alpha + 2\beta(q+r) + \gamma(qq+rr) + 2\delta qr + 2\varepsilon qr(q+r) + \zeta qqrr = 0$$
$$\alpha + 2\beta(r+s) + \gamma(rr+ss) + 2\delta rs + 2\varepsilon rs(r+s) + \zeta rrss = 0$$
etc.

donec ad vltimam z perueniatur.

4.) Relatio quae hinc concluditur inter p et z, erit integrale completum aequationis differentialis propositae, et littera a vicem gerit constantis arbitrariae per integrationem ingressae.

Corol-

Corollarium.

643. Hinc etiam integrale completum inueniri poteſt huius aequationis differentialis

$$\frac{m\,\partial y}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} = \frac{n\,\partial x}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}},$$

deſignantibus m et n numeros integros. Statuatur enim vtrumque membrum $= \frac{\partial u}{\sqrt{(A+2Bu+Cuu+2Du^3+Eu^4)}}$, et quaeratur relatio tam inter x et u, quam inter y et u; vnde eliſa u orietur aequatio algebraica inter x et y.

Scholion.

644. Ne hic extractio radicis in ſingulis aequationibus repetenda ambiguitatem creet, loco vniuscuiusque vti conveniet binis per extractionem iam erutis. Scilicet vt ex prima valor q rite per p definiatur, primo quidem habemus

$$q = \frac{-\beta-\delta p-\varepsilon pp+2\sqrt{\Delta(A+2Bp+Cpp+2Dp^3+Ep^4)}}{\gamma+2\varepsilon p+\zeta pp},$$

tum vero capi debet

$$2\sqrt{\Delta(A+2Bq+Cqq+2Dq^3+Eq^4)} = -\beta-\delta q-\varepsilon qq-p(\gamma+2\varepsilon q+\zeta qq):$$

ſimilique modo in relatione inter binas ſequentes quantitates inueſtiganda erit procedendum. Caeterum adhuc notari convenit numeros integros m et n poſitiuos eſſe debere, neque hanc inueſtigationem ad negatiuos extendi, propterea quod formula differentialis $\frac{\partial z}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$, poſito z negatiuo, naturam ſuam mutat. Interim tamen cum hanc aequalitatem

$$\Pi:x+\Pi:y = \text{Conſt.}$$

ſupra algebraice expreſſerimus, eius ope quoque ii caſus reſolui poſſunt, vbi eſt m vel n numerus negatiuus: ſi enim fuerit

$$\Pi:z = n\,\Pi:p + \text{Conſt.}$$

quaeratur y, vt fit

$$\Pi : y + \Pi : z = \text{Conft.}$$

eritque

$$\Pi : y = -n\,\Pi : p + \text{Conft.}$$

Problema 84.

645. Si $\Pi : z$ eiusmodi functionem transcendentem ipfius z denotet, vt fit

$$\Pi : z = \int \frac{\partial z(\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}zz + \mathfrak{D}z^3 + \mathfrak{E}z^4)}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}},$$

comparationem inter huiusmodi functiones inueftigare.

Solutio.

Ex coefficientibus A, B, C, D, E, vna cum conftante arbitraria L determinentur fequentes valores

$\alpha = 4(AC - BB + AL)$, $\beta = 4AD + 2BL$, $\gamma = 4AE - LL$,
$\zeta = 4(CE - DD + EL)$, $\varepsilon = 4BE + 2DL$, $\delta = 4AE + 4BD + 2CL + LL$,
et inter binas variabiles x et y haec conftituatur relatio

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy = 0,$$

eritque

$$\frac{\partial x}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} + \frac{\partial y}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = 0,$$

pro qua fine ambiguitate habetur

$$\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx) = 2\sqrt{\Delta(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}$$

$$\beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy) = 2\sqrt{\Delta(A + 2By + Cyy + 2Dy^3 + Ey^4)}$$

exiftente

$$\Delta = L^3 + CL^2 + 4(BD - AE)L + 4(ADD + BBE - ACE).$$

Quare fi ponamus

$$\frac{\partial x(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4)}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} + \frac{\partial y(\mathfrak{A} + \mathfrak{B}y + \mathfrak{C}y^2 + \mathfrak{D}y^3 + \mathfrak{E}y^4)}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = 2\partial V\sqrt{\Delta},$$

vt fit

$\Pi : x$

$$\Pi : x + \Pi : y = \text{Const.} + 2\,V\sqrt{\Delta}, \text{ erit}$$

$$\frac{\partial x\,[\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y)]}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} = 2\,\partial V\sqrt{\Delta}, \text{ seu}$$

$$\partial V = \frac{\partial x\,[\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y^4)]}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}.$$

Ponatur nunc $x+y=t$ et $xy=u$, et quia $\partial x+\partial y=\partial t$ et $x\partial y+y\partial x=\partial u$, erit $\partial x=\frac{x\partial t-\partial u}{x-y}$, seu $(x-y)\partial x = x\partial t-\partial u$, tum vero est $x=\frac{1}{2}t+\sqrt{(\frac{1}{4}tt-u)}$. At his positionibus aequatio assumta induit hanc formam

$$\alpha+2\beta t+\gamma tt+2(\delta-\gamma)u+2\varepsilon tu+\zeta uu=0,$$

vnde fit differentiando

$$\partial t(\beta+\gamma t+\varepsilon u)+\partial u(\delta-\gamma+\varepsilon t+\zeta u)=0, \text{ ergo}$$

$$\partial t=\frac{-\partial u(\delta-\gamma+\varepsilon t+\zeta u)}{\beta+\gamma t+\varepsilon u}, \text{ et}$$

$$x\partial t-\partial u=\frac{-\partial u\,[\beta+\gamma t+\varepsilon u+(\delta-\gamma)x+\varepsilon tx+\zeta ux]}{\beta+\gamma t+\varepsilon u} \text{ siue}$$

$$x\partial t-\partial u=\frac{-\partial u\,[\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)]}{\beta+\gamma t+\varepsilon u}$$

sicque habebimus

$$\frac{\partial x(x-y)}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}=\frac{-\partial u}{\beta+\gamma t+\varepsilon u}; \text{ ergo}$$

$$\partial V=\frac{-\partial u\,[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{\beta+\gamma t+\varepsilon u} \text{ seu}$$

$$\partial V=\frac{+\partial t\,[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{\delta-\gamma+\varepsilon t+\zeta u}.$$

Est vero aequatione illa resoluta.

$$t=\frac{-\beta-\varepsilon u+\sqrt{[\beta\beta-\alpha\gamma+2(\gamma\gamma+\beta\varepsilon-\gamma\delta)u+(\varepsilon\varepsilon-\gamma\zeta)uu]}}{\gamma} \text{ seu}$$

$$t=\frac{-\beta-\varepsilon u+2\sqrt{\Delta(A+Lu+Euu)}}{\gamma},$$

vnde conficitur

$$\partial V=\frac{-\partial u\,[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{2\sqrt{\Delta(A+Lu+Euu)}};$$

ideoque

$$\Pi : x+\Pi : y=\text{Const.}-\int\frac{\partial u\,[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{\sqrt{(A+Lu+Euu)}}.$$

Vel cum reperiatur

$u=$

$$u = \frac{-(\delta-\gamma)-\varepsilon t+\sqrt{[(\delta-\gamma)^2-\alpha\zeta+2[(\delta-\gamma)\varepsilon-\beta\zeta)t+(\varepsilon\varepsilon-\gamma\zeta)tt]}}{\zeta}$$

quae expreſſio abit in hanc

$$u = \frac{-(\delta-\gamma)-\varepsilon t+2\sqrt{\Delta(L+2Dt+Ett)}}{\zeta}$$

vnde fit

$$\partial V = \frac{\partial t\,[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{2\sqrt{\Delta(L+C+2Dt+Ett)}}$$

ſicque habebimus per t

$$\Pi : x + \Pi : y = \text{Conſt.} + \int \frac{\partial t\,[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{\sqrt{(L+C+2Dt+Ett)}},$$

quae expreſſio, niſi ſit algebraica, certe vel per logarithmos, vel arcus circulares exhiberi poteſt. Tum vero poſt integrationem tantum opus eſt, vt loco t reſtituatur eius valor $x+y$.

Corollarium 1.

646. Si velimus, vt poſito $x=a$ ſiat $y=b$, conſtans L ita debet definiri, vt ſit

$$\tfrac{1}{2}L(b-a)^2 = A + B(a+b) + Cab + Dab(a+b) + Eaabb$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bb+Cbb+2Db^3+Eb^4)},$$

tum igitur conſtans noſtra erit $= \Pi : a + \Pi : b$, integrali poſtremo ita ſumto, vt euaneſcat poſito $t = a+b$.

Corollarium 2.

647. Eodem modo etiam differentia functionum $\Pi : x - \Pi : y$ exprimi poteſt, mutando alterutrius formulae radicalis ſignum, quo pacto formularum differentialium ſignum alterius conuertetur.

Corollarium 3.

648. Quantitas V comparationi harum functionum inſeruiens, erit algebraica, ſi haec formula differentialis

$$\frac{\partial t\,[\mathfrak{B}+\mathfrak{C}\zeta t+\mathfrak{D}(\delta-\gamma+\varepsilon t+\zeta tt)+\mathfrak{E}[2(\delta-\gamma)+2\varepsilon t+\zeta tt]t]}{\zeta\sqrt{(L+C+2Dt+Ett)}}$$

inte-

integrationem admittat; quia altera pars $\frac{-2\partial t\sqrt{\Delta}}{\zeta}(\mathfrak{D}+2\mathfrak{E}t)$ per se est integrabilis.

Scholion.

649. Hoc ergo argumentum plane nouum de comparatione huiusmodi functionum transcendentium tam copiose pertractauimus, quam praesens institutum postulare videbatur. Quando autem eius applicatio ad comparationem arcuum curvarum, quorum longitudo huiusmodi functionibus exprimitur, erit facienda, vberiori euolutione erit opus, vbi contemplatio singularium proprietatum, quae hoc modo eruuntur, eximium vsum afferre poterit. Commode autem hoc argumentum ad doctrinam de resolutione aequationum differentialium referri videtur, siquidem inde eiusmodi aequationum integralia completa et quidem algebraice exhiberi possunt, quae aliis methodis frustra indagantur. Hunc igitur huic sectionis finem faciet methodus generalis omnium aequationum differentialium integralia proxime determinandi.

CA-

CAPVT VII.

DE
INTEGRATIONE AEQVATIONVM DIFFERENTIALIVM PER APPROXIMATIONEM.

Problema 85.

650.

Propofita aequatione differentiali quacunque, eius integrale completum vero proxime affignare.

Solutio.

Sint x et y binae variabiles, inter quas aequatio differentialis proponitur, atque haec aequatio huiusmodi habebit formam vt fit $\frac{\partial y}{\partial x} = V$, exiftente V functione quacunque ipfarum x et y. Iam cum integrale completum defideretur, hoc ita eft interpretandum, vt dum ipfi x certus quidem valor puta $x = a$ tribuitur, altera variabilis y datum quemdam valorem puta $y = b$ adipifcatur. Quaeftionem ergo primo ita tractemus, vt inueftigemus valorem ipfius y, quando ipfi x valor paulisper ab a difcrepans tribuitur, feu pofito $x = a + \omega$, vt quaeramus y. Cum autem ω fit particula minima, etiam valor ipfius y minime a b difcrepabit; vnde dum x ab a vsque ad $a + \omega$ tantum mutatur, quantitatem V interea tanquam conftantem fpectare licet. Quare pofito $x = a$ et $y = b$ fiat $V = A$, et pro hac exigua mutatione habebimus $\frac{\partial y}{\partial x} = A$, ideoque integrando $y = b + A(x - a)$, eiusmodi fcilicet conftante adiecta, vt pofito $x = a$ fiat $y = b$. Statuamus ergo $x = a + \omega$, fiet-

fietque $y = b + A\omega$. Quemadmodum ergo hic ex valoribus initio datis $x = a$ et $y = b$, proxime sequentes $x = a + \omega$ et $y = b + A\omega$ inuenimus, ita ab his simili modo per interualla minima vlterius progredi licet, quoad tandem ad valores a primitiuis quantumuis remotos perueniatur. Quae operationes quo clarius ob oculos ponantur, sequenti modo successiue instituantur.

Ipsius	valores successiui
x	$a, a', a'', a''', a^{IV}, \ldots \ldots {}'x, x$
y	$b, b', b'', b''', b^{IV}, \ldots \ldots {}'y, y$
V	$A, A', A'', A''', A^{IV}, \ldots \ldots {}'V, V.$

Scilicet ex primis $x = a$ et $y = b$ datis, habetur $V = A$; tum vero pro secundis erit $b' = b + A(a' - a)$, differentia $a' - a$ minima pro lubitu assumta. Hinc ponendo $x = a'$ et $y = b'$ colligitur $V = A'$, indeque pro tertiis obtinebitur $b'' = b' + A'(a'' - a')$, vbi posito $x = a''$ et $y = b''$ inuenitur $V = A''$. Iam pro quartis habebimus $b''' = b'' + A''(a''' - a'')$, hincque ponendo $x = a'''$ et $y = b'''$, colligemus $V = A'''$, sicque ad valores a primitiuis quantumuis remotos progredi licebit. Series autem prima valores ipsius x successiuos exhibens pro lubitu accipi potest, dummodo per interualla minima ascendat vel etiam descendat.

Corollarium 1.

651. Pro singulis ergo interuallis minimis calculus eodem modo instituitur, sicque valores, a quibus sequentia pendent, obtinentur. Hoc ergo modo singulis pro x assumtis valoribus, valores respondentes ipsius y assignari possunt.

Corollarium 2.

652. Quo minora accipiuntur interualla, per quae valores ipsius x progredi assumuntur, eo accuratius valores pro

singulis eliciuntur. Interim tamen errores in singulis commissi, etiamsi sint multo minores, ob multitudinem coaceruantur.

Corollarium 3.

653. Errores autem in hoc calculo inde oriuntur, quod in singulis interuallis ambas quantitates x et y vt constantes spectemus, sicque functio V pro constante habeatur. Quo magis ergo valor ipsius V a quouis interuallo ad sequens immutatur, eo maiores errores sunt pertimescendi.

Scholion 1.

654. Hoc incommodum imprimis occurrit, vbi valor ipsius V vel euanescit vel in infinitum excrescit, etiamsi mutationes ipsis x et y accidentes sint satis paruae. His autem casibus errores saltim enormes sequenti modo euitabuntur: sit pro initio huiusmodi interualli $x = a$ et $y = b$, tum vero in ipsa aequatione proposita ponatur $x = a + \omega$ et $y = b + \psi$, vt sit $\frac{\partial \psi}{\partial \omega} = V$, in V autem ita fiat substitutio $x = a + \omega$ et $y = b + \psi$, vt quantitates ω et ψ tanquam minimae spectentur, reiiciendo scilicet altiores potestates prae inferioribus, hoc enim modo plerumque integratio pro his interuallis actu institui poterit. Hac autem emendatione vix vnquam erit opus, nisi termini ex ipsis valoribus a et b nati se destruant. Veluti si habeatur haec aequatio $\frac{\partial y}{\partial x} = \frac{aa}{xx - yy}$, ac pro initio debeat esse $x = a$ et $y = a$; iam pro interuallo hinc incipiente ponatur $x = a + \omega$ et $y = a + \psi$ habebiturque $\frac{\partial \psi}{\partial \omega} = \frac{aa}{2a\omega - 2a\psi}$, seu $2\omega\partial\psi - 2\psi\partial\psi = a\partial\omega$, seu $\partial\omega - \frac{2\omega\partial\psi}{a} = \frac{-2\psi\partial\psi}{a}$, quae per $e^{\frac{-2\psi}{a}} = 1 - \frac{2\psi}{a}$ multiplicata et integrata praebet

$$\left(1 - \frac{2\psi}{a}\right)\omega = \frac{-2}{a}\int\left(1 - \frac{2\psi}{a}\right)\psi\partial\psi = -\frac{\psi\psi}{a},$$

quia posito $\omega = 0$ fieri debet $\psi = 0$. Hinc ergo habetur $\omega =$

$\omega = \frac{-\psi\psi}{a - 2\psi} = \frac{-\psi\psi}{a}$, seu $a(a' - a) = -(b' - b)^2$, existente $b = a$; vnde colligitur pro sequente interuallo $b' = b + \sqrt{-a(a' - a)}$, quo casu patet valorem x non vltra a augeri posse, quia y fieret imaginarium.

Scholion 2.

655. Passim traduntur regulae aequationum differentialium integralia per series infinitas exprimendi, quae autem plerumque hoc vitio laborant, vt integralia tantum particularia exhibeant, praeterquam quod series illae certo tantum casu conuergant, neque ergo aliis casibus vllum vsum praestent. Veluti si proposita sit aequatio $\partial y + y \partial x = a x^n \partial x$, iubemur huiusmodi seriem in genere fingere

$$y = A x^\alpha + B x^{\alpha+1} + C x^{\alpha+2} + D x^{\alpha+3} + E x^{\alpha+4} + \text{etc.}$$

qua substituta fit

$$\left.\begin{array}{l} \alpha A x^{\alpha-1} + (\alpha+1) B x^\alpha + (\alpha+2) C x^{\alpha+1} + (\alpha+3) D x^{\alpha+2} + \text{etc.} \\ \quad\cdot \quad + \quad A \quad + \quad B \quad + \quad C \quad + \text{etc.} \\ -a x^n \end{array}\right\} = 0$$

Statuatur ergo $\alpha - 1 = n$, seu $\alpha = n + 1$, eritque $A = \frac{a}{n+1}$, tum vero reliquis terminis ad nihilum reductis

$$B = \frac{-A}{n+2}, \; C = \frac{-B}{n+3}, \; D = \frac{-C}{n+4}, \text{ etc.}$$

sicque habebitur haec series

$$y = \frac{a x^{n+1}}{n+1} - \frac{a x^{n+2}}{(n+1)(n+2)} + \frac{a x^{n+3}}{(n+1)(n+2)(n+3)} - \frac{a x^{n+4}}{(n+1)(n+2)(n+3)(n+4)} \text{ etc.}$$

Verum hoc integrale tantum est particulare, quoniam euanescente x, simul y euanescit, nisi $n + 1$ sit numerus negatiuus; tum vero haec series non conuergit, nisi x capiatur valde par-

 vum.

vum. Quamobrem hinc minime cognofcere licet valores ipfius y, qui refpondeant valoribus quibuscunque ipfius x. Hoc autem vitio non laborat methodus, quam hic adumbrauimus, cum primo integrale completum praebeat, dum fcilicet pro dato ipfius x valore datum ipfi y valorem tribuit, tum vero per interualla minima procedens, femper proxime ad veritatem accedat, et quousque libuerit progredi liceat. Sequenti autem modo haec methodus magis perfici poterit.

Problema 86.

656. Methodum praecedentem, aequationes differentiales proxime integrandi, magis perficere, vt minus a veritate aberret.

Solutio.

Propofita aequatione integranda $\frac{\partial y}{\partial x} = V$, error methodi fupra expofitae inde oritur, quod per fingula interualla functio V vt conftans fpectetur, cum tamen reuera mutationem fubeat, praecipue nifi interualla ftatuantur minima. Variabilitas autem ipfius V per quoduis interuallum fimili modo in computum duci poteft, quo in fectione praecedente §. 321. vfi fumus. Scilicet fi iam ipfi x conueniat y, ex natura differentialium ipfi $x - n\partial x$ vidimus conuenire

$$y - n\partial y + \frac{n(n+1)}{1.2}\partial\partial y - \frac{n(n+1)(n+2)}{1.2.3}\partial^3 y + \text{etc.}$$

qui valor fumto n infinito erit

$$y - n\partial y + \frac{nn\partial\partial y}{1.2} - \frac{n^3\partial^3 y}{1.2.3} + \frac{n^4\partial^4 y}{1.2.3.4} - \text{etc.}$$

Statuatur iam $x - n\partial x = a$ et

$$y - n\partial y + \frac{nn\partial\partial y}{1.2} - \frac{n^3\partial^3 y}{1.2.3} + \frac{n^4\partial^4 y}{1.2.3.4} - \text{etc.} = b,$$

hicque valores in quouis interuallo vt primi fpectentur, dum extremi per x et y indicantur. Cum igitur fit $n = \frac{x-a}{\partial x}$, fiet

$y =$

$$y = b + \frac{(x-a)\partial y}{\partial x} - \frac{(x-a)^2 \partial\partial y}{1.2\,\partial x^2} + \frac{(x-a)^3 \partial^3 y}{1.2.3\,\partial x^3} - \frac{(x-a)^4 \partial^4 y}{1.2.3.4\,\partial x^4} + \text{etc.}$$

quae expreſſio, ſi x non multum ſuperat a, valde conuergit, ideoque admodum eſt idonea ad valorem y proxime inueniendum. Verum ad ſingulos terminos huius ſeriei euoluendos, notari oportet eſſe $\frac{\partial y}{\partial x} = V$, hincque $\frac{\partial\partial y}{\partial x^2} = \frac{\partial V}{\partial x}$. Cum autem V ſit functio ipſarum x et y, ſi ponamus $\partial V = M\partial x + N\partial y$, ob $\frac{\partial y}{\partial x} = V$, erit $\frac{\partial\partial y}{\partial x^2} = M + NV$, ſeu exprimendi modo iam ſupra expoſito $\frac{\partial\partial y}{\partial x^2} = (\frac{\partial V}{\partial x}) + V(\frac{\partial V}{\partial y})$, quae expreſſio vti nata eſt ex praecedente $\frac{\partial y}{\partial x} = V$, ita ex ea naſcetur ſequens

$$\frac{\partial^3 y}{\partial x^3} = (\frac{\partial\partial V}{\partial x^2}) + (\frac{\partial V}{\partial x})(\frac{\partial V}{\partial y}) + 2V(\frac{\partial\partial V}{\partial x\partial y}) + V(\frac{\partial V}{\partial y})^2 + VV(\frac{\partial\partial V}{\partial y^2}).$$

Quoniam vero ipſe valor ipſius y nondum eſt cognitus, hoc modo ſaltem obtinetur aequatio algebraica, qua relatio inter x et y exprimitur; niſi forte ſufficiat in terminis poſuiſſe $y = b$.

Altera autem operatio §. 322. expoſita valorem ipſius y, qui ipſi x in fine cuiusque interualli reſpondet, explicite determinabit, cum in initio eiusdem interualli fuerit $x = a$ et $y = b$. Cum enim hinc poſito $x = a + n\partial a$, ſi quidem a et b vt variabiles ſpectemus, fiat

$$y = b + n\partial b + \frac{n(n-1)}{1.2}\partial\partial b + \frac{n(n-1)(n-2)}{1.\;2.\;3}\partial^3 b + \text{etc.}$$

quia eſt $n = \frac{x-a}{\partial a}$, ideoque numerus infinitus, erit

$$y = b + \frac{(x-a)\partial b}{\partial a} + \frac{(x-a)^2 \partial\partial b}{1.2\,\partial a^2} + \frac{(x-a)^3 \partial^3 b}{1.2.3\,\partial a^3} + \text{etc.}$$

Eſt vero $\frac{\partial b}{\partial a} = V$, ſiquidem in functione V ſcribatur $x = a$ et $y = b$; tum vero iisdem pro x et y valoribus ſubſtitutis, erit

$$\frac{\partial\partial b}{\partial a^2} = (\frac{\partial V}{\partial x}) + V(\frac{\partial V}{\partial y}) \text{ et}$$

$$\frac{\partial^3 b}{\partial a^3} = (\frac{\partial\partial V}{\partial x^2}) + 2V(\frac{\partial\partial V}{\partial x\partial y}) + VV(\frac{\partial\partial V}{\partial y^2}) + (\frac{\partial V}{\partial y})[(\frac{\partial V}{\partial x}) + V(\frac{\partial V}{\partial y})],$$

vnde ſequentes ſimili modo formari oportet. Sit igitur poſt-

quam, ſcripſerimus $x = a$ et $y = b$,

$$\frac{\partial y}{\partial x} = A,\ \frac{\partial \partial y}{\partial x^2} = B,\ \frac{\partial^3 y}{\partial x^3} = C,\ \frac{\partial^4 y}{\partial x^4} = D, \text{ etc.}$$

ac valori $x = a + \omega$ conueniet iſte valor

$$y = b + A\omega + \tfrac{1}{2}B\omega^2 + \tfrac{1}{6}C\omega^3 + \tfrac{1}{24}D\omega^4 + \text{ etc.}$$

qui duo valores iam pro ſequente interuallo erunt initiales; ex quibus ſimili modo finales erui oportet.

Corollarium 1.

657. Quoniam hic variabilitatis functionis V rationem habuimus, interualla iam maiora ſtatuere licet, ac ſi illas formulas A, B, C, D, etc. in infinitum continuare vellemus, interualla quantumuis magna aſſumi poſſent, tum autem pro y oriretur ſeries infinita.

Corollarium 2.

658. Si ſeriei inuentae tantum binos terminos primos ſumamus, vt ſit $y = b + A\omega$, habebitur determinatio praecedens, vnde ſimul patet errorem ibi commiſſum ſequentibus terminis iunctim ſumtis aequari.

Corollarium 3.

659. Etiamſi autem ſeriei inuentae plures terminos capiamus, conſultum tamen non erit interualla nimis magna conſtitui, vt ω valorem modicum obtineat, praecipue ſi quantitates B, C, D, etc. euadant valde magnae.

Scholion.

660. Maximo incommodo hae operationes turbantur, ſi quando horum coefficientium A, B, C, D, etc. quidam in infinitum excreſcant. Euenit autem hoc tantum in certis interuallis, vbi ipſa quantitas V vel in nihilum abit vel in infini-

finitum, cui incommodo, quaemadmodum fit occurrendum, iam innuimus et mox accuratius oftendemus. Caeterum calculus pro fingulis interuallis pari modo inftituitur, ita vt cum eius ratio pro interuallo primo fuerit inuenta, quod incipit a valoribus pro lubitu affumtis $x = a$ et $y = b$, eadem pro fequentibus interuallis fit valitura. Cum enim pro fine interualli primi fiat

$$x = a + \omega = a' \text{ et}$$

$$y = b + A\omega + \tfrac{1}{2}B\omega^2 + \tfrac{1}{6}C\omega^3 + \tfrac{1}{24}D\omega^4 + \text{etc.} = b',$$

hi erunt valores initiales pro interuallo fecundo, ex quibus fimili modo finales elici oportet; hic fcilicet calculus innitetur perinde litteris a' et b', ac prior litteris a et b, id quod clarius ex exemplis fubiunctis patebit.

Exemplum I.

661. *Aequationis differentialis* $\partial y = \partial x\,(x^n + cy)$ *integrale completum proxime inueftigare.*

Cum hic fit $V = \frac{\partial y}{\partial x} = x^n + cy$, erit differentiando

$$\frac{\partial\partial y}{\partial x^2} = n x^{n-1} + c x^n + c c y,$$

ficque porro

$$\frac{\partial^3 y}{\partial x^3} = n(n-1)x^{n-2} + n c x^{n-1} + c c x^n + c^3 y$$

$$\frac{\partial^4 y}{\partial x^4} = n(n-1)(n-2)x^{n-3} + n(n-1)c x^{n-2} + ncc x^{n-1} + c^3 x^n + c^4 y$$

etc.

Quodfi ergo ponamus valori $x = a$, conuenire $y = b$, alii cuicunque valori $x = a + \omega$ conueniet

$$y = b + \omega(cb + a^n) + \tfrac{1}{2}\omega^2(ccb + c a^n + n a^{n-1})$$

$$+ \tfrac{1}{6}\omega^3[c^3 b + c c a^n + n c a^{n-1} + n(n-1)a^{n-}]$$

$$+ \tfrac{1}{24}\omega^4[c^4 b + c^3 a^n + ncc a^{n-1} + n(n-1)c a^{n-2} + n(n-1(n-2)a^{n-3}]$$

etc.

quae

quae feries fumta quantitate ω fatis parua, quantumuis promte conuergit, ficque pofito $a+\omega = a'$ et refpondente valore ipfius $y = b'$, hinc fimili modo ad fequentes perueniemus, quam operationem, quousque lubuerit, continuare licet.

Exemplum 2.

662. *Aequationis differentialis* $\partial y = \partial x\,(xx + yy)$ *integrale completum proxime inuestigare*

Cum hic fit $\frac{\partial y}{\partial x} = V = xx + yy$, erit continuo differentiando

$$\frac{\partial\partial y}{\partial x^2} = 2x + 2xxy + 2y^3 \text{ et}$$

$$\frac{\partial^3 y}{\partial x^3} = 2 + 4xy + 2x^4 + 8xxyy + 6y^4$$

$$\frac{\partial^4 y}{\partial x^4} = 4y + 12x^3 + 20xyy + 16x^4y + 40xxy^3 + 24y^5$$

$$\frac{\partial^5 y}{\partial x^5} = 40x^2 + 24y^2 + 104x^3y + 120xy^3 + 16x^6 + 136x^4y^2 + 240x^2y^4 + 120y^6.$$

Quare fi initio fit $x = a$ et $y = b$, erit

$$A = aa + bb$$

$$B = 2a + 2aab + 2b^3$$

$$C = 2 + 4ab + 2a^4 + 8aabb + 6b^4$$

$$D = 4b + 12a^3 + 20abb + 16a^4b + 40aab^3 + 24b^5$$

$$E = 40a^2 + 24b^2 + 104a^3b + 120ab^3 + 16a^6 + 136a^4b^2 + 240a^2b^4 + 120b^6,$$

vnde valori cuicunque alii $x = a + \omega$ conueniet

$$y = b + A\omega + \tfrac{1}{2}B\omega^2 + \tfrac{1}{6}C\omega^3 + \tfrac{1}{24}D\omega^4 + \tfrac{1}{120}E\omega^5 + \text{etc.}$$

atque ex talibus binis valoribus, qui fint $x = a'$ et $y = b'$, denuo fequentes elici poffunt.

Scho-

Scholion.

663. Quoniam totum negotium ad inuentionem horum coefficientium A, B, C, D, etc. redit, obferuo eosdem fine differentiatione inueniri poffe, id quod in hoc poftremo exemplo $\frac{\partial y}{\partial x} = xx + yy$ ita praeftabitur. Cum ftatuamus pofito $x = a$ fieri $y = b$, ponamus in genere $x = a + \omega$ et $y = b + \psi$, et noftra aequatio induet hanc formam

$$\frac{\partial \psi}{\partial \omega} = aa + bb + 2a\omega + \omega\omega + 2b\psi + \psi\psi$$

et quia euanefcente ω fimul euanefcit ψ, fumamus

$$\psi = \alpha\omega + \beta\omega^2 + \gamma\omega^3 + \delta\omega^4 + \varepsilon\omega^5 + \text{ etc.}$$

hocque valore fubftituto prodibit

$$\begin{aligned} \alpha + 2\beta\omega + 3\gamma\omega^2 + 4\delta\omega^3 + 5\varepsilon\omega^4 + \text{ etc.} = \\ aa + bb + 2a\omega + \omega\omega \qquad\qquad\qquad\qquad \\ + 2\alpha b\omega + 2\beta b\omega^2 + 2\gamma b\omega^3 + 2\delta b\omega^4 + \text{ etc.} \\ + \alpha^2\omega^2 \quad + 2\alpha\beta\omega^3 + 2\alpha\gamma\omega^4 + \text{ etc.} \\ + \beta\beta\omega^4 \end{aligned}$$

fingulis ergo terminis ad nihilum reductis fiet

$$\alpha = aa + bb,\ 2\beta = 2\alpha b + 2a,\ 3\gamma = 2\beta b + \alpha\alpha + 1,$$
$$4\delta = 2\gamma b + 2\alpha\beta,\ 5\varepsilon = 2\delta b + 2\alpha\gamma + \beta\beta,$$
$$6\zeta = 2\varepsilon b + 2\alpha\delta + 2\beta\gamma, \text{ etc.}$$

vnde iidem valores qui fupra per differentiationem eliciuntur. Vti haec methodus fimplicior eft praecedente, ita etiam hoc illi praeftat, quod femper in vfum vocari poffit, cum illa interdum fruftra applicetur, veluti in exemplis allatis euenit, fi valores initiales a et b euanefcant, vbi plerique coefficientes in nihilum abirent. Quod idem incommodum iam fupra animaduertimus, cum adeo euenire poffit, vt omnes coefficientes vel euanefcant, vel in infinitum abeant. Verum hoc nonnifi

in certis interuallis vſu venit, pro quibus ergo calculum peculari modo inſtitui conueniet; reliquis autem interuallis methodus hic expoſita per differentiationem procedens commodius adhiberi videtur, quippe quae ſaepe facilius inſtituitur quam ſubſtitutio, certisque regulis continetur, ſemper locum habentibus etiam in aequationibus tranſcendentibus. Quare pro ſingularibus illis interuallis praecepta tradere oportebit.

Problema 87.

664. Si in integratione aequationis $\frac{\partial y}{\partial x} = V$ pro quopiam interuallo eueniat, vt quantitas V vel euaneſcat, vel fiat infinita, integrationem pro iſto interuallo inſtituere.

Solutio.

Sit pro initio interualli, quod contemplamur $x = a$ et $y = b$, quo caſu cum V vel euaneſcat vel in infinitum abeat, ponamus $\frac{\partial y}{\partial x} = \frac{P}{Q}$, ita vt poſito $x = a$ et $y = b$, vel P vel Q vel vtrumque euaneſcat. Statuamus ergo vt ab his terminis vlterius progrediamur, $x = a + \omega$ et $y = b + \psi$, fietque $\frac{\partial y}{\partial x} = \frac{\partial \psi}{\partial \omega}$; atque tam P quam Q erit functio ipſarum ω et ψ, quarum altera ſaltem euaneſcat, facto $\omega = 0$ et $\psi = 0$. Iam ad rationem inter ω et ψ proxime ſaltem inueſtigandam, ponatur $\psi = m\,\omega^n$, erit $\frac{\partial \psi}{\partial \omega} = m\,n\,\omega^{n-1}$, hincque $m\,n\,Q\,\omega^{n-1} = P$; vbi P et Q ob $\psi = m\,\omega^n$ meras poteſtates ipſius ω continebunt, quarum tantum minimas in calculo retinuiſſe ſufficit, cum altiores prae his vt euaneſcentes ſpectari queant. Infimae ergo poteſtates ipſius ω inter ſe aequales reddantur, ſimulque ad nihilnm redigantur; vnde tam exponens n quam coefficiens m determinabitur. Si deinde relationem inter ω et ψ exactius cognoſcere velimus, inuentis m et n, ad altiores poteſtates aſcendamus ponendo

$\psi =$

$$\psi = m\,\omega^n + M\,\omega^{n+\mu} + N\,\omega^{n+\nu}\ \text{etc.}$$

hincque ſimili modo ſequentes partes definientur, quousque ob magnitudinem interualli ſeu particulae ω neceſſarium viſum fuerit.

Corollarium 1.

665. Si poſito $x = a$ et $y = b$, neque P neque Q euaneſcat, ſubſtitutione adhibita reperietur $\frac{\partial \psi}{\partial \omega} = \frac{A + etc.}{\alpha + etc.}$, hincque proxime $\alpha\,\partial\psi = A\,\partial\omega$ et $\psi = \frac{A}{\alpha}\,\omega$, qui eſt primus terminus praecedentis approximationis, quo inuento reliqui vt ante ſe habebunt.

Corollarium 2.

666. Si α tantum euaneſcat, habebitur

$$\frac{\partial \psi}{\partial \omega}(M\,\omega^\mu + N\,\psi^\nu\ \text{etc.}) = A$$

proxime: vnde poſito $\psi = m\,\omega^n$ fit

$$A = m\,n\,\omega^{n-1}(M\,\omega^\mu + N\,m^\nu\,\omega^{n\nu});$$

quod autem non valet, niſi ſit $\nu(1-\mu) > \mu$ ſeu $\nu > \frac{\mu}{1-\mu}$. Sin autem ſit $\nu < \frac{\mu}{1-\mu}$, ſtatui debet $n - 1 + n\,\nu = 0$ ſeu $n = \frac{1}{1+\nu}$, altero termino vt infima poteſtate ſpectata. At ſi fuerit $\nu = \frac{\mu}{1-\mu}$, ambo termini pro paribus poteſtatibus erunt habendi, fietque $n = 1 - \mu$ et $A = m\,n\,(M + N\,m^\nu)$, vnde m definiri debet.

Scholion.

667. In genere hic vix quicquam praecipere licet, ſed quouis caſu oblato haud difficile eſt omnia, quae ad ſolutionem perducunt perſpicere. Si quidem omnes exponentes eſſent integri, regula illa *Neutoniana*, qua ope parallelogrammi reſolutio aequationum inſtruitur, hic in vſum vocari poſ-

ſet; tum vero exponentium fractorum ad integros reductio ſatis eſt nota. Verum huiusmodi caſus tam raro occurrunt, vt inutile foret in praeceptis prolixum eſſe, quae quouis caſu ab exercitato facile conduntur. Veluti ſi perueniatur ad hanc aequationem $\frac{\partial\psi}{\partial\omega}(\alpha\sqrt{\omega}+\beta\psi)=\gamma$, ex ſuperioribus patet primam operationem dare $\psi=m\sqrt{\omega}$, vnde fit $\frac{1}{2}m(\alpha+\beta m)=\gamma$, vnde m innoteſcit idque duplici modo. Quin etiam haec aequatio, poſito $\sqrt{\omega}=p$, ad homogeneitatem reducitur, ideoque reuera integrari poteſt: verum haec vix vnquam vſum habitura fuſius non proſequor, ſed, quod adhuc in hac parte pertractandum reſtat exponam, quomodo eiusmodi aequationes differentiales reſolui oporteat, in quibus differentialium ratio puta $\frac{\partial y}{\partial x}=p$ vel plures obtinet dimenſiones, vel adeo tranſcendenter ingreditur, quo abſoluto partem ſecundam, in qua differentialia altiorum graduum occurrunt, aggrediar.

CAL-

CALCVLI INTEGRALIS
LIBER PRIOR.

PARS PRIMA,

SEV

METHODVS INVESTIGANDI FVNCTIONES VNIVS VARIABILIS EX DATA RELATIONE QVACVNQVE DIFFERENTIALIVM PRIMI GRADVS.

SECTIO TERTIA,

DE

RESOLVTIONE AEQVATIONVM DIFFERENTIALIVM MAGIS COMPLICATARVM.

DE

RESOLVTIONE AEQVATIONVM DIFFERENTIALIVM IN QVIBVS DIFFERENTIALIA AD PLVRES DIMENSIONES ASSVRGVNT, VEL ADEO TRANSCENDENTER IMPLICANTVR.

Problema 88.

668.

Posita differentialium relatione $\frac{\partial y}{\partial x} = p$, si proponatur aequatio quaecunque inter binas quantitates x et p, relationem inter ipsas variabiles x et y inuestigare.

Solutio.

Cum detur aequatio inter p et x, concessa aequationum resolutione, ex ea quaeratur p per x, ac reperietur functio ipsius x, quae ipsi p erit aequalis. Peruenietur ergo ad huiusmodi aequationem $p = X$, existente X functione quapiam ipsius x tantum. Quare cum sit $p = \frac{\partial y}{\partial x}$, habebimus $\partial y = X \partial x$, sicque quaestio ad sectionem primam est reducta, vnde formulae $X \partial x$ integrale inuestigari oportet; quo facto integrale quaesitum erit $y = \int X \partial x$.

Si aequatio inter x et p data, ita fuerit comparata, vt inde facilius x per p definiri possit, quaeratur x, prodeatque $x = P$, existente P functione quadam ipsius p. Hac igitur aequatione differentiata erit $\partial x = \partial P$, hincque $\partial y = p \partial x = p \partial P$, vnde integrando elicitur $y = \int p \partial P$, seu $y = pP - \int P \partial p$. Hinc ergo ambae variabiles x et y per tertiam p ita determinantur, vt sit

$x = P$

$x = P$ et $y = pP - \int P \partial p$,

vnde relatio inter x et y eſt manifeſta.

Si neque p commode per x, neque x per p definiri queat, ſaepe effici poteſt, vt vtraque commode per nouam quantitatem u definiatur; ponamus ergo inueniri $x = U$ et $p = V$, vt U et V ſint functiones eiusdem variabilis u. Hinc ergo erit $\partial y = p\,\partial x = V\,\partial U$, et $y = \int V\,\partial U$, ſicque x et y per eandem nouam variabilem u exprimuntur.

Corollarium 1.

669. Simili modo reſoluetur caſus, quo aequatio quaecunque inter p et alteram variabilem y proponitur, quoniam binas variabiles x et y inter ſe permutare licet. Tum autem ſiue p per y, ſiue y per p, ſiue vtraque per nouam variabilem u definiatur, notari oportet, eſſe $\partial x = \frac{\partial y}{p}$.

Corollarium 2.

670. Cum $\surd(\partial x^2 + \partial y^2)$ exprimat elementum arcus curuae, cuius coordinatae rectangulae ſunt x et y, ſi ratio

$$\frac{\surd(\partial x^2 + \partial y^2)}{\partial x} = \surd(1 + pp), \text{ ſeu } \frac{\surd(\partial x^2 + \partial y^2)}{\partial y} = \frac{\surd(1 + pp)}{p},$$

aequetur functioni vel ipſius x vel ipſius y, hinc relatio inter x et y inueniri poterit.

Corollarium. 3.

671. Quoniam hoc modo relatio inter x et y per integrationem inuenitur, ſimul noua quantitas conſtans introducitur, quocirca illa relatio pro integrali completo erit habenda.

Scholion 1.

672. Hactenus eiusmodi tantum aequationes differentiales examini ſubiicimus, quibus poſito $\frac{\partial y}{\partial x} = p$, eiusmodi relatio

latio inter ternas quantitates x, y et p proponitur, vnde valor ipsius p commode per x et y exprimi potest, ita vt $p = \frac{\partial y}{\partial x}$ aequetur functioni cuipiam ipsarum x et y. Nunc igitur eiusmodi relationes inter x, y et p considerandae veniunt, ex quibus valorem ipsius p vel minus commode, vel plane non, per x et y definire liceat; atque hic simplicissimus casus sine dubio est, quando in relatione proposita altera variabilis x seu y plane deest, ita vt tantum relatio inter p et x vel p et y proponatur; quem casum in hoc problemate expediuimus. Solutionis autem vis in eo versatur, vt proposita aequatione inter x et p, non littera p per x, nisi forte hoc facile praestari queat, sed potius x per p, vel etiam vtraque per nouam variabilem u definiatur. Veluti si proponatur haec aequatio

$$x\,\partial x + a\,\partial y = b\sqrt{(\partial x^2 + \partial y^2)},$$

quae posito $\frac{\partial y}{\partial x} = p$, abit in hanc

$$x + ap = b\sqrt{(1 + pp)},$$

hinc minus commode definiretur p per x. Cum autem sit

$$x = b\sqrt{(1+pp)} - ap, \text{ ob } y = \int p\,\partial x = px - \int x\,\partial p,$$

erit

$$y = bp\sqrt{(1+pp)} - app - b\int \partial p\sqrt{(1+pp)} + \tfrac{1}{2}app;$$

sicque relatio inter x et y constat. Sin autem peruentum fuerit ad talem aequationem

$$x^3\,\partial x^3 + \partial y^3 = ax\,\partial x^2\,\partial y \text{ seu } x^3 + p^3 = apx,$$

hic neque x per p neque p per x commode definire licet; ex quo pono $p = ux$, vnde fit $x + u^3 x = au$, hincque $x = \frac{au}{1+u^3}$ et $p = \frac{auu}{1+u^3}$. Iam ob $\partial x = \frac{a\,\partial u(1-2u^3)}{(1+u^3)^2}$, colligitur $y = aa\int\frac{uu\,\partial u(1-2u^3)}{(1+u^3)^3}$, ac reducendo hanc formam ad simpliciorem

$y =$

$$y = \tfrac{1}{6}\,a\,a\,.\,\frac{2\,u^3 - 1}{(1 + u^3)^2} - a\,a\int\frac{u\,u\,\partial u}{(1 + u^3)^2} \text{ seu}$$
$$y = \tfrac{1}{6}\,a\,a\,.\,\frac{2\,u^3 - 1}{(1 + u^3)^2} + \tfrac{1}{3}\,a\,a\,.\,\frac{1}{1 + u^3} + \text{Const.}$$

Scholion 2.

673. Cum igitur hunc casum, quo aequatio vel inter x et p vel inter y et p proponitur, generatim expedire licuerit, videndum est quibus casibus euolutio succedat, quando omnes tres quantitates x, y et p in aequatione proposita insunt. Ac primo quidem obseruo, dummodo binae variabiles x et y vbique eundem dimensionum numerum adimpleant, quomodocunque praeterea quantitas p ingrediatur, resolutionem semper ad casus ante tractatos reuocari posse; tales scilicet aequationes perinde tractare licet, atque aequationes homogeneas, ad quod genus etiam merito referuntur, cum dimensiones a differentialibus natae vbique debeant esse pares, et iudicium ex solis quantitatibus finitis x et y peti oporteat. Quae ergo dummodo vbique eundem dimensionum numerum constituant, aequatio pro homogenea erit habenda, veluti est

$$x\,x\,\partial y - y\,y\,\sqrt{(\partial x^2 + \partial y^2)} = 0 \text{ seu}$$
$$p\,x\,x - y\,y\,\sqrt{(1 + p\,p)} = 0.$$

Deinde etiam eiusmodi aequationes euolutionem admittunt, in quibus altera variabilis x vel y plus vna dimensione nusquam habet, vtcunque praeterea differentialium ratio $p = \frac{\partial y}{\partial x}$ ingrediatur. Hos ergo casus hic accuratius explicemus.

Problema 89.

674. Posito $p = \frac{\partial y}{\partial x}$, si in aequatione inter x, y et p proposita, binae variabiles x et y vbique eundem dimensionum numerum compleant, inuenire relationem inter x et y, quae illius aequationis sit integrale completum.

So-

Solutio.

Cum in aequatione inter x, y et p propofita binae variabiles x et y vbique eundem dimenfionum numerum conftituant, fi ponamus $y = u x$, quantitas x inde per diuifionem tolletur, habebiturque aequatio inter duas tantum quantitates u et p, qua earum relatio ita definietur, vt vel u per p, vel p per u determinari poffit. Iam ex pofitione $y = u x$ fequitur $\partial y = u \partial x + x \partial u$, cum igitur fit $\partial y = p \partial x$, erit $p \partial x - u \partial x = x \partial u$, ideoque $\frac{\partial x}{x} = \frac{\partial u}{p - u}$. Quia itaque p per u datur, formula differentialis $\frac{\partial u}{p - u}$ vnicam variabilem complectens per regulas primae fectionis integretur, eritque $l x = \int \frac{\partial u}{p - u}$, ficque x per u determinatur; et cum fit $y = u x$, ambae variabiles x et y per eandem tertiam variabilem u determinantur, et quia illa integratio conftantem arbitrariam inducit, haec relatio inter x et y erit integrale completum.

Corollarium 1.

675. Cum fit $\frac{\partial x}{x} = \frac{\partial u}{p - u}$, erit etiam $l x = - l (p - u) + \int \frac{\partial p}{p - u}$, quae formula commodior eft, fi forte ex aequatione inter p et u propofita, quantitas u facilius per p definitur.

Corollarium 2.

676. Quodfi integrale $\int \frac{\partial u}{p - u}$ vel $\int \frac{\partial p}{p - u}$ per logarithmos exprimi poffit, vt fit $\int \frac{\partial u}{p - u} = l\,\mathrm{U}$, erit $l x = l\,\mathrm{C} + l\,\mathrm{U}$; hincque $x = \mathrm{C\,U}$, et $y = \mathrm{C\,U}\,u$; vnde relatio inter x et y algebraice dabitur: et cum fit $u = \frac{y}{x}$, haec tertia variabilis u facile eliditur.

Scholion.

677. Eandem hanc refolutionem fupra in aequationibus homogeneis ordinariis docuimus, quae ergo ob dimenfio-

nes differentialium non turbatur; quin etiam fuccedit, etiamfi ratio differentialium $\frac{\partial y}{\partial x} = p$ tranfcendenter ingrediatur. Hoc modo fcilicet refolutio ad integrationem aequationis differentialis feparatae $\frac{\partial x}{x} = \frac{\partial u}{p - u}$ perducitur, quemadmodum etiam fupra per priorem methodum negotium fuit expeditum. Altera vero methodus, qua fupra vfi fumus, quaerendo factorem qui aequationem differentialem reddat per fe integrabilem, hic plane locum non habet, cum per differentiationem aequationis finitae nunquam differentialia ad plures dimenfiones exfurgere queant. Non ergo hoc modo inuenitur aequatio finita inter x et y, quae differentiata ipfam aequationem propofitam reproducat, fed quae faltem cum ea conueniat, et quidem non obftante arbitraria illa conftante, quae per integrationem ingreffa, integrale completum reddit.

Exemplum 1.

678. *Si in aequationem propofitam neutra variabilium x et y ipfa ingrediatur, fed tantum differentialium ratio $\frac{\partial y}{\partial x} = p$, integrale completum affignare.*

Pofito ergo $\frac{\partial y}{\partial x} = p$, aequatio propofita folam variabilem p cum conftantibus complectetur, vnde ex eius refolutione, prout plures inuoluat radices, orietur $p = \alpha$, $p = \beta$, $p = \gamma$ etc. Iam ob $p = \frac{\partial y}{\partial x}$, ex fingulis radicibus integralia completa elicientur, quae erunt

$$y = \alpha x + a,\ y = \beta x + b,\ y = \gamma x + c, \text{ etc.}$$

quae fingula aequationi propofitae aeque fatisfaciunt. Quae fi velimus omnia vna aequatione finita complecti, erit integrale completum

$$(y - \alpha x - a)(y - \beta x - b)(y - \gamma x - c) \text{ etc.} = 0,$$

quae vti apparet non vnam nouam conftantem, fed plures a, b, c

b, c, etc. comprehendit, tot ſcilicet, quot aequatio differentialis plurium dimenſionum habuerit radices.

Corollarium 1.

679. Ita aequationis differentialis $\partial y^2 - \partial x^2 = 0$ ſeu $pp - 1 = 0$, ob $p = +1$ et $p = -1$, duo habemus integralia $y = x + a$ et $y = -x + b$, quae in vnum collecta dant $(y - x - a)(y + x - b) = 0$, ſeu

$$yy - xx - (a + b)y - (a - b)x + ab = 0.$$

Corollarium 2.

680. Propoſita aequatione $\partial y^3 + \partial x^3 = 0$ ſeu $p^3 + 1 = 0$, ob radices $p = -1$, $p = \frac{1 + \sqrt{-3}}{2}$, et $p = \frac{1 - \sqrt{-3}}{2}$, erit vel $y = -x + a$, vel $y = \frac{1 + \sqrt{-3}}{2} x + b$, vel $y = \frac{1 - \sqrt{-3}}{2} x + c$, quae collecta prebent

$$\begin{aligned} y^3 + x^3 - (a + b + c)yy + \left(a - \tfrac{1 - \sqrt{-3}}{2} b - \tfrac{1 + \sqrt{+3}}{2} c\right) xy \\ + \left(-a + \tfrac{1 - \sqrt{-3}}{2} b + \tfrac{1 + \sqrt{-3}}{2} c\right) xx + (ab + ac + bc) y \\ + \left(bc - \tfrac{1 - \sqrt{-3}}{2} ac - \tfrac{1 + \sqrt{-3}}{2} ab\right) x - abc = 0, \end{aligned}$$

quae aequatio etiam ita exhiberi poteſt

$$y^3 + x^3 - fyy - gxy - hxx + Ay + Bx + C = 0,$$

vbi conſtantes A, B, C, ita debent eſſe comparatae, vt aequatio haec reſolutionem in tres ſimplices admittat.

Exemplum 2.

681. *Propoſita aequatione differentiali*

$$y \partial x - x \sqrt{(\partial x^2 + \partial y^2)} = 0,$$

eius integrale completum inuenire.

Poſito $\frac{\partial y}{\partial x} = p$, fit $y - x\sqrt{(1 + pp)} = 0$; ſit ergo $y = ux$, erit $u = \sqrt{(1 + pp)}$, et $\frac{\partial x}{x} = \frac{\partial u}{p - u}$, vnde per alte-

 ram

ram formulam

$$lx = -l(p-u) + \int \frac{\partial p}{p - \surd(1+pp)} = -l(p-u) - \int \partial p [p + \surd(1+pp)]$$

at

$$\int \partial p \surd(1+pp) = \tfrac{1}{2} p \surd(1+pp) + \tfrac{1}{2} l [p + \surd(1+pp)];$$

vnde colligitur

$$lx = C - \tfrac{1}{2} l [\surd(1+pp) - p] - \tfrac{1}{2} p \surd(1+pp) - \tfrac{1}{2} pp$$

vel

$$lx = C + \tfrac{1}{2} l [\surd(1+pp) + p] - \tfrac{1}{2} p \surd(1+pp) - \tfrac{1}{2} pp, \text{ et}$$

$$y = ux = x \surd(pp + 1).$$

Exemplum 3.

682. *Huius aequationis*

$$y \partial x - x \partial y = nx \surd(\partial x^2 + \partial y^2)$$

integrale completum inuenire.

Ob $\frac{\partial y}{\partial x} = p$, nostra aequatio est $y - px = nx\surd(1+pp)$, quae posito $y = ux$, abit in $u - p = n\surd(1+pp)$. Cum ergo sit

$$lx = -l(p-u) + \int \frac{\partial p}{p-u}, \text{ erit}$$

$$lx = -l n \surd(1+pp) - \int \frac{\partial p}{n\surd(1+pp)},$$

hincque

$$lx = C - l n \surd(1+pp) - \tfrac{1}{n} l [p + \surd(1+pp)].$$

Quare habetur

$$x = \frac{a}{\surd(1+pp)} [\surd(1+pp) - p]^{\frac{1}{n}}, \text{ et}$$

$$y = \frac{a[p + n\surd(1+pp)]}{\surd(1+pp)} [\surd(1+pp) - p]^{\frac{1}{n}}.$$

Cum nunc sit $uu - 2up + pp = nn + nnpp$, erit

$$p = \frac{u - n\surd(uu+1-nn)}{1-nn} \text{ et } \surd(1+pp) = \frac{-nu + \surd(uu+1-nn)}{1-nn}$$

atque

$$\sqrt{(1+pp)}-p=\frac{-u+\sqrt{(uu+1-nn)}}{1-n},$$

vnde fit

$$\frac{x[-nu+\sqrt{(uu+1-nn)}]}{a(1-nn)}=\left(\frac{-u+\sqrt{(uu+1-nn)}}{1-n}\right)^{\frac{1}{n}}, \text{ vbi } u=\frac{y}{x}.$$

At si $n=1$, erit $p=\frac{uu-1}{2u}$, $\sqrt{(1+pp)}=\frac{uu+1}{2u}$, atque $x=\frac{2au}{uu+1}\cdot\frac{1}{u}=\frac{2axx}{xx+yy}$, seu $yy+xx=2ax$.

Si $n=-1$, est quidem vt ante

$$p=\frac{uu-1}{2u}, \text{ et } \sqrt{(1+pp)}=\frac{-uu-1}{2u},$$

vnde

$$x=\frac{a}{\sqrt{(1+pp)}}[\sqrt{(1+pp)}+p]=\frac{-2a}{1+uu}=\frac{-2axx}{xx+yy}.$$

Ergo et $x=0$, et $xx+yy+2ax=0$.

Scholion.

683. Haec aequatio sumendis vtrinque quadratis et radice $p=\frac{\partial y}{\partial x}$ extrahenda, ad aequationem homogeneam ordinariam reducitur. Fit enim primo

$$yy-2pxy+ppxx=nnxx+nnppxx,$$

tum vero

$$px=\frac{x\partial y}{\partial x}=\frac{y\pm n\sqrt{(yy+xx-nnxx)}}{1-nn},$$

quae posito $y=ux$ separabilis redditur. Vbi imprimis casus quo $nn=1$ notari meretur, quo fit $yy-2pxy=xx$, seu $p=\frac{\partial y}{\partial x}=\frac{yy-xx}{2xy}$, ideoque

$$2xy\partial y+xx\partial x-yy\partial x=0:$$

quae etiam per partes integrari potest, cum $2xy\partial y-yy\partial x$ integrabile fiat per factorem $\frac{1}{xyy}f:\frac{yy}{x}$, quo vt etiam pars $xx\partial x$ integrabilis reddatur, illa forma abit in $\frac{1}{xx}$, sicque habebitur

$$\frac{2xy\partial y-yy\partial x}{xx}+\partial x=0,$$

cuius integrale est $\frac{yy}{x}+x=2a$, vt ante, nisi quod altera solu-

solutio $x = 0$ hinc non eliciatur. Verum cum aequatio illa quadrata posito $n = 1$, subito abeat in simplicem, altera radix perit, quae reperitur ponendo $n = 1 - \alpha$, quo fit

$$yy - 2pxy = xx - 2\alpha xx - 2\alpha ppxx,$$

ideoque px infinitum, reiectis ergo terminis prae reliquis euanescentibus est $-pxy = xx - 2\alpha ppxx$, quae diuisibilis per x, alteram praebet solutionem $x = 0$. Talis quidem resolutio succedit, quando valorem p per radicis extractionem elicere licet; sed si aequatio ad plures dimensiones ascendat, vel adeo transcendens fiat, methodo hic exposita carere non possumus.

Exemplum 4.

684. *Proposita aequatione*

$$x\,\partial y^3 + y\,\partial x^3 = \partial y\,\partial x\,\sqrt{xy(\partial x^2 + \partial y^2)},$$

eius integrale completum inuestigare.

Posito $\frac{\partial y}{\partial x} = p$, et $y = ux$, nostra aequatio induet hanc formam $p^3 + u = p\sqrt{u(1+pp)}$, vnde conficitur

$$\frac{\partial x}{x} = \frac{\partial u}{p-u}, \text{ seu } lx = \int \frac{\partial u}{p-u} = -l(p-u) + \int \frac{\partial p}{p-u}.$$

Inde autem est

$$\sqrt{u} = \tfrac{1}{2}p\sqrt{(1+pp)} + \tfrac{1}{2}p\sqrt{(1-4p+pp)},$$

et quadrando

$$u = \tfrac{1}{2}pp - p^3 + \tfrac{1}{2}p^4 + \tfrac{1}{2}pp\sqrt{(1+pp)(1-4p+pp)},$$

hincque

$$p - u = \tfrac{1}{2}p(1+pp)(2-p) - \tfrac{1}{2}pp\sqrt{(1+pp)(1-4p+pp)},$$

vnde colligimus

$$\frac{\partial p}{p-u} = \frac{\partial p(2-p)}{2p(1-p+pp)} + \frac{\partial p\sqrt{(1-4p+pp)}}{2(1-p+pp)\sqrt{(1+pp)}}.$$

In quorum membrorum posteriore, si ponatur $\sqrt{\frac{1-4p+pp}{1+pp}} = q$, ob

ob $p = \frac{2 + \sqrt{[4 - (1 - qq)^2]}}{1 - qq}$, $\partial p = \frac{4 q \partial q [2 + \sqrt{(4 - (1 - qq)^2)}]}{(1 - qq)^2 \sqrt{[4 - (1 - qq)^2]}}$, et

$$1 - p + pp = \frac{(3 + qq)[2 + \sqrt{(4 - (1 - qq)^2)}]}{(1 - qq)^2},$$

obtinebitur

$$\int \frac{\partial p}{p - u} = \frac{1}{2} \int \frac{\partial p (2 - p)}{p (1 - p + pp)} + 2 \int \frac{qq \partial q}{(3 + qq) \sqrt{[4 - (1 - qq)^2]}},$$

vbi membrum posterius neque per logarithmos, neque arcus circulares integrari potest.

Exemplum 5.

685. *Inuenire relationem inter* x et y, *vt posito* $s = \int \sqrt{(\partial x^2 + \partial y^2)}$, *fiat* $ss = 2xy$.

Cum sit $s = \sqrt{2xy}$, erit

$$\partial s = \sqrt{(\partial x^2 + \partial y^2)} = \frac{x \partial y + y \partial x}{\sqrt{2xy}},$$

hincque posito $\frac{\partial y}{\partial x} = p$ et $y = ux$, fiet $\sqrt{(1 + pp)} = \frac{p + u}{\sqrt{2u}}$, seu $u = \sqrt{2u(1 + pp)} - p$, et radice extracta

$$\sqrt{u} = \sqrt{\frac{1 + pp}{2}} + \frac{1 - p}{\sqrt{2}} = \frac{1 - p + \sqrt{(1 + pp)}}{\sqrt{2}},$$

quare

$$u = 1 - p + pp + (1 - p)\sqrt{(1 + pp)}, \text{ et}$$

$$p - u = -(1 - p)[1 - p + \sqrt{(1 + pp)}].$$

Ergo

$$\int \frac{\partial p}{p - u} = \int \frac{\partial p}{2p(1 - p)} [1 - p - \sqrt{(1 + pp)}] = \frac{1}{2} l p - \frac{1}{2} \int \frac{\partial p \sqrt{(1 + pp)}}{p (1 - p)}.$$

At posito $p = \frac{1 - qq}{2q}$, fit

$$\int \frac{\partial p \sqrt{(1 + pp)}}{p (1 - p)} = \int \frac{-\partial q (1 + qq)^2}{q (1 - qq)(qq + 2q - 1)} = + \int \frac{\partial q}{q} - 2 \int \frac{\partial q}{1 - qq} - 4 \int \frac{\partial q}{(q + 1)^2 - 2}$$

$$= + l q - l \frac{1 + q}{1 - q} + \sqrt{2}\, l \frac{\sqrt{2} + 1 + q}{\sqrt{2} - 1 - q},$$

hincque

$$\int \frac{\partial p}{p - u} = \frac{1}{2} l p - \frac{1}{2} l q + \frac{1}{2} l \frac{1 + q}{1 - q} - \frac{1}{\sqrt{2}} l \frac{\sqrt{2} + 1 + q}{\sqrt{2} - 1 - q}$$

$$= l \left(\frac{1 + q}{2q}\right) - \frac{1}{\sqrt{2}} l \frac{\sqrt{2} + 1 + q}{\sqrt{2} - 1 - q}.$$

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$$p - u = \frac{(1+q)(1-2q-qq)}{2q} = + \frac{(1+q)[2-(1+q)^2]}{2q},$$

ſicque habetur

$$lx = C - l(1+q) + lq - l[2-(1+q)^2] + l\left(\frac{1+q}{q}\right)$$
$$- \frac{1}{\sqrt{2}}\, l\, \frac{\sqrt{2}+1+q}{\sqrt{2}-1-q} = la - l[2-(1+q)^2] - \frac{1}{\sqrt{2}}\, l\, \frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$

vbi eſt $u = \frac{y}{x} = \frac{1}{2}(1+q)^2$, et $1 + q = \sqrt{\frac{2y}{x}}$, vnde

$$x = \frac{ax}{x-y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}} \text{ ſeu } x - y = a\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}, \text{ vel}$$

$$(\sqrt{x}+\sqrt{y})^{1+\frac{1}{\sqrt{2}}} = a(\sqrt{x}-\sqrt{y})^{\frac{1}{\sqrt{2}}-1}.$$

Eſt ergo aequatio inter x et y interſcendens, vti vocari ſolet.

Scholion.

686. Facilius haec reſolutio abſoluitur quaerendo ſtatim ex aequatione

$$u + p = \sqrt{2u(1+pp)}, \text{ ſeu } uu + 2up + pp = 2u + 2upp$$

valorem ipſius p, qui fit

$$p = \frac{u + \sqrt{(uu - 4uu + 2u + 2u^3 - uu)}}{2u-1}, \text{ ſeu } p = \frac{u + (1-u)\sqrt{2u}}{2u-1}, \text{ et}$$

$$p - u = \frac{(1-u)(2u+\sqrt{2u})}{2u-1} = \frac{(1-u)\sqrt{2u}}{\sqrt{2u}-1}.$$

Quare

$$lx = \int \frac{\partial u}{p-u} = \int \frac{\partial u(\sqrt{2u}-1)}{(1-u)\sqrt{2u}} = C - l(1-u) - \int \frac{\partial u}{(1-u)\sqrt{2u}}.$$

Sit $u = vv$, eritque

$$\int \frac{\partial u}{(1-u)\sqrt{2u}} = \frac{1}{\sqrt{2}} \int \frac{2\partial v}{1-vv} = \frac{1}{\sqrt{2}}\, l\, \frac{1+v}{1-v},$$

hincque

$$lx = la - l(1-u) - \frac{1}{\sqrt{2}}\, l\, \frac{1+\sqrt{u}}{1-\sqrt{u}}.$$

Vnde ob $u = \frac{y}{x}$, reperitur $x = \frac{ax}{x-y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}$, vt ante. Quare ſi curua deſideretur coordinatis rectangulis x et y determi-

terminanda, vt eius arcus s fit $= \surd 2xy$, erit aequatio eius naturam definiens

$$(\surd x + \surd y)^{\frac{1}{\surd 2} + 1} = a(\surd x - \surd y)^{\frac{1}{\surd 2} - 1}.$$

Caeterum euidens eft fimili modo quaeftionem refolui poffe, fi arcus s functioni cuicunque homogeneae vnius dimenfionis ipfarum x et y aequaetur, feu fi proponatur aequatio quaecunque homogenea inter x, y et s, id quod fequenti problemate oftendiffe operae erit pretium.

Problema 90.

687. Si fuerit $s = \int \surd(\partial x^2 + \partial y^2)$, atque aequatio proponatur homogenea quaecunque inter x, y et s, in qua fcilicet hae tres variabiles x, y et s, vbique eundem dimenfionum numerum conftituant, inuenire aequationem finitam inter x et y.

Solutio.

Ponatur $y = ux$ et $s = vx$, vt hac fubftitutione ex aequatione homogenea propofita variabilis x elidatur, et aequatio obtineatur inter binas u et v, vnde v per u definiri poffit. Tum vero fit $\partial y = p\,\partial x$, eritque

$$\partial s = \partial x \surd(1 + pp), \text{ vnde fit}$$

$$p\,\partial x = u\,\partial x + x\,\partial u, \text{ et } \partial x \surd(1 + pp) = v\,\partial x + x\,\partial v,$$

ergo

$$\frac{\partial x}{x} = \frac{\partial u}{p - u} = \frac{\partial v}{\surd(1 + pp) - v}.$$

Quia nunc v datur per u, fit $\partial v = q\,\partial u$, vt habeatur

$$\surd(1 + pp) = v + pq - qu,$$

et fumtis quadratis

$$1 + pp = (v - qu)^2 + 2pq(v - qu) + ppqq,$$

vnde elicitur

 $p =$

$$p = \frac{q(v-qu)+\sqrt{[(v-qu)^2-1+qq]}}{1-qq} \text{ et}$$
$$p-u = \frac{qv-u+\sqrt{[(v-qu)^2-1+qq]}}{1-qq}.$$

Quare hinc deducimus

$$\frac{\partial x}{x} = \frac{\partial u(1-qq)}{qv-u+\sqrt{[(v-qu)^2-1+qq]}} = \frac{\partial u[qv-u-\sqrt{[(v-qu)^2-1+qq]}}{1+uu-vv},$$

vnde cum v et q detur per u, inueniri poteſt x per eandem u: at ob $q\partial u = \partial v$ fiet

$$lx = la - l\sqrt{(1+uu-vv)} - \int\frac{\partial u\sqrt{[(v-qu)^2-1+qq]}}{1+uu-vv},$$

tum vero eſt $y = ux$, ſeu poſito $\frac{y}{x}$ loco u habebitur aequatio quaeſita inter x et y.

Corollarium 1.

688. Cum s exprimat arcum curuae coordinatis rectangulis x et y reſpondentem, ſic definitur curua, cuius arcus aequatur functioni cuicunque vnius dimenſionis ipſarum x et y; quae ergo erit algebraica, ſi integrale

$$\int\frac{\partial u\sqrt{[(v-qu)^2-1+qq]}}{1+uu-vv}$$

per logarithmos exhiberi poteſt.

Corollarium 2.

689. Simili modo reſolui poterit problema, ſi s eiusmodi formulam integralem exprimat, vt ſit $\partial s = Q\partial x$, exiſtente Q functione quacunque quantitatum p, u et v. Tum autem ex aequalitate $\frac{\partial x}{x} = \frac{\partial u}{p-u} = \frac{\partial u}{Q-v}$ valorem ipſius p elici oportet, et quia v per u datur, erit $lx = \int\frac{\partial u}{p-u}$.

Exemplum 1.

690. Si debeat eſſe $s = \alpha x + \beta y$, erit $v = \alpha + \beta u$, et $q = \frac{\partial v}{\partial u} = \beta$, hinc $v - qu = \alpha$, ergo

$$lx = la - l\sqrt{[1+uu-(\alpha+\beta u)^2]} - \int\frac{\partial u\sqrt{(\alpha\alpha+\beta\beta-1)}}{1+uu-(\alpha+\beta u)^2},$$

quae

quae postrema pars est

$$-\int\frac{\partial u\sqrt{(\alpha\alpha+\beta\beta-1)}}{1-\alpha\alpha-2\alpha\beta u+(1-\beta\beta)uu}=(\alpha\alpha+\beta\beta-1)^{\frac{1}{2}}\int\frac{\partial u}{\alpha\alpha-1+2\alpha\beta u+(\beta\beta-1)uu},$$

quae transformatur in

$$\int\frac{(\beta\beta-1)\partial u\sqrt{(\alpha\alpha+\beta\beta-1)}}{[u(\beta\beta-1)+\alpha\beta-\sqrt{(\alpha\alpha+\beta\beta-1)}]\,[u(\beta\beta-1)+\alpha\beta+\sqrt{(\alpha\alpha+\beta\beta-1)}]}$$
$$=\frac{1}{2}\,l\,\frac{(\beta\beta-1)u+\alpha\beta-\sqrt{(\alpha\alpha+\beta\beta-1)}}{(\beta\beta-1)u+\alpha\beta+\sqrt{(\alpha\alpha+\beta\beta-1)}}.$$

Quare posito $u=\frac{y}{x}$, aequatio integralis quaesita est, sumtis quadratis,

$$\frac{xx+yy-(\alpha x+\beta y)^2}{\alpha\alpha}=\frac{(\beta\beta-1)y+\alpha\beta x-x\sqrt{(\alpha\alpha+\beta\beta-1)}}{(\beta\beta-1)y+\alpha\beta x+x\sqrt{(\alpha\alpha+\beta\beta-1)}}.$$

At posito

$$(\beta\beta-1)y+\alpha\beta x-x\sqrt{(\alpha\alpha+\beta\beta-1)}=P$$
$$(\beta\beta-1)y+\alpha\beta x+x\sqrt{(\alpha\alpha+\beta\beta-1)}=Q$$

est

$$PQ=(\beta\beta-1)^2yy+2\alpha\beta(\beta\beta-1)xy+(\alpha\alpha-1)(\beta\beta-1)xx$$
$$=(\beta\beta-1)[(\alpha x+\beta y)^2-xx-yy],$$

vnde mutata constante fit $\frac{PQ}{bb}=\frac{P}{Q}$, ergo vel $P=0$ vel $Q=b$; solutio ergo in genere est

$$(\beta\beta-1)y+\alpha\beta x\pm x\sqrt{(\alpha\alpha+\beta\beta-1)}=c,$$

quae est aequatio pro linea recta.

Exemplum 2.

691. Si debeat esse $s=\frac{nyy}{x}$, erit $v=nuu$ et $q=2nu$; vnde $1+uu-vv=1+uu-nnu^4$ et $v-qu=-nuu$, ergo

$$lx=la-l\sqrt{(1+uu-nnu^4)}-\int\frac{\partial u\sqrt{(nnu^4-1+4nnuu)}}{1+uu-nnu^4}$$

quae formula autem per logarithmos integrari nequit.

Exemplum 3.

692. Si debeat esse $ss = xx + yy$, erit $v = \sqrt{(1+uu)}$ et $q = \frac{u}{\sqrt{(1+uu)}}$, vnde fit $1 + uu - vv = 0$, solutionem ergo ex primis formulis repeti conuenit, vnde fit

$$v - qu = \frac{1}{\sqrt{(1+uu)}},$$

$$qq - 1 = \frac{-1}{1-uu}, \text{ et } qv - u = 0;$$

ergo $p - u = 0$, seu $\frac{\partial y}{\partial x} - \frac{y}{x} = 0$, ita vt prodeat $y = nx$.

Exemplum 4.

693. Si debeat esse $ss = yy + nxx$, seu $v = \sqrt{(uu+n)}$, et $q = \frac{u}{\sqrt{(uu+n)}}$, erit $1 + uu - vv = 1 - n$; $v - qu = \frac{n}{\sqrt{(uu+n)}}$ et $qq - 1 = \frac{-n}{uu+n}$. Quare habebitur

$$lx = la - l\sqrt{(1-n)} - \frac{1}{1-n}\int\frac{\partial u\sqrt{(nn+n)}}{\sqrt{(uu+n)}}$$

$$= lb + \frac{\sqrt{n}}{\sqrt{(n-1)}}\, l\,[u + \sqrt{(uu+n)}],$$

hincque

$$\frac{x}{b} = \left(\frac{y + \sqrt{(yy+nxx)}}{x}\right)^{\sqrt{\frac{n-1}{n}}}.$$

Quoties ergo $\frac{n}{n-1}$ est numerus quadratus, aequatio inter x et y prodit algebraica. Sit $\sqrt{\frac{n}{n-1}} = m$, erit $n = \frac{mm}{mm-1}$ et $ss = yy + \frac{mmxx}{mm-1}$, cui conditioni satisfit hac aequatione algebraica

$$x^{m+1} = b\,[y + \sqrt{(yy + \frac{mmxx}{mm-1})}]^m$$

quae tranformatur in

$$x^{\frac{2}{m}} - 2b^{\frac{1}{m}}x^{\frac{1-m}{m}}y = \frac{mm}{mm-1}b^{\frac{2}{m}}, \text{ seu}$$

$$y = \frac{(mm-1)x^{\frac{2}{m}} - mmb^{\frac{2}{m}}}{2(mm-1)b^{\frac{2}{m}}x^{\frac{1-m}{m}}}$$

Corol-

Corollarium.

694. Ponamus $m = \frac{1}{n}$, ac si fuerit

$$y = \frac{b^{2n} + (nn - 1) x^{2n}}{2 (nn - 1) b^n x^{n-1}}, \text{ erit}$$

$$ss = yy - \frac{xx}{nn - 1}, \text{ seu } s = \sqrt{(yy - \frac{xx}{nn - 1})}.$$

Quare si

$$y = \frac{b^4 + 3 x^4}{6 b b x}, \text{ est } s = \sqrt{(yy - \frac{xx}{3})}.$$

Problema 91.

695. Si posito $\frac{\partial y}{\partial x} = p$, eiusmodi detur aequatio inter x, y et p, in qua altera variabilis y vnicam tantum habeat dimensionem, inuenire relationem inter binas variabiles x et y.

Solutio.

Hinc ergo y aequabitur functioni cuipiam ipsarum x et p, vnde differentiando fiet $\partial y = P \partial x + Q \partial p$. Cum igitur sit $\partial y = p \partial x$, habebitur haec aequatio differentialis $(P - p) \partial x + Q \partial p = 0$, quam integrari oportet. Quoniam tantum duas continet variabiles x et p, et differentialia simpliciter inuoluit, eius resolutio per methodos supra expositas est tentanda.

Primo ergo resolutio succedet, si fuerit $P = p$, ideoque $\partial y = p \partial x + Q \partial p$. Quod euenit, si y per x et p ita determinetur, vt sit $y = p x + \Pi$, denotante Π functionem quamcunque ipsius p. Tum ergo erit $Q = x + \frac{\partial \Pi}{\partial p}$, et cum solutio ab ista aequatione $Q \partial p = 0$ pendeat, erit vel $\partial p = 0$, hincque $p = \alpha$, seu $y = \alpha x + \beta$, vbi altera constantium α et β per ipsam aequationem propositam determinatur, dum posito $p = \alpha$ fit $\beta = \Pi$; vel erit $Q = 0$, ideoque $x = - \frac{\partial \Pi}{\partial p}$, et $y = - \frac{p \partial \Pi}{\partial p} + \Pi$, vbi

vbi ergo vtraque solutio est algebraica, si modo Π fuerit functio algebraica ipsius p.

Secundo, aequatio $(P-p)\partial x + Q\partial p = 0$, resolutionem admittet, si altera variabilis x cum suo differentiali ∂x vnam dimensionem non superet. Euenit hoc si fuerit $y = Px + \Pi$, dum P et Π sunt functiones ipsius p tantum, tum enim erit $P = P$ et $Q = \frac{x\partial P}{\partial p} + \frac{\partial \Pi}{\partial p}$, hincque haec habeatur aequatio integranda

$$(P-p)\partial x + x\partial P + \partial\Pi = 0 \text{ seu } \partial x + \frac{x\partial P}{P-p} = -\frac{\partial\Pi}{P-p},$$

quae per $e^{\int\frac{\partial P}{P-p}}$ multiplicata dat

$$e^{\int\frac{\partial P}{P-p}}x = -\int e^{\int\frac{\partial P}{P-p}}\frac{\partial\Pi}{P-p}.$$

Siue ponatur $\frac{\partial P}{P-p} = \frac{\partial R}{R}$, erit aequatio integralis

$$Rx = C - \int\frac{R\partial\Pi}{P-p} = C - \int\frac{\partial\Pi\,\partial R}{\partial P};$$

vnde fit

$$x = \frac{C}{R} - \frac{1}{R}\int\frac{\partial\Pi\,\partial R}{\partial P}, \text{ et}$$

$$y = \frac{CP}{R} + \Pi - \frac{P}{R}\int\frac{\partial\Pi\,\partial R}{\partial P}.$$

Tertio resolutio nullam habebit difficultatem, si denotantibus X et V functiones quascunque ipsius x, fuerit $y = X + Vp$. Tum enim erit

$$\partial y = p\partial x = \partial X + V\partial p + p\partial V,$$

ideoque

$$\partial p + p\left(\frac{\partial V - \partial x}{V}\right) = -\frac{\partial X}{V},$$

sit $\frac{\partial x}{V} = \frac{\partial R}{R}$, vt R sit etiam functio ipsius x, erit

$$\frac{V}{R}p = C - \int\frac{\partial X}{R}, \text{ seu } p = \frac{CR}{V} - \frac{R}{V}\int\frac{\partial X}{R}, \text{ et}$$

$y =$

$$y = X + CR - R\int\frac{\partial X}{R},$$

quae aequatio relationem inter x et y exprimit.

Quarto aequatio $(P - p)\,\partial x + Q\,\partial p = 0$ resolutionem admittit si fuerit homogenea. Cum ergo terminus $p\,\partial x$ duas contineat dimensiones, hoc euenit, si totidem dimensiones et in reliquis terminis insint. Vnde perspicuum est, P et Q esse debere functiones homogeneas vnius dimensionis ipsarum x et p. Quare si y ita per x et p definiatur, vt y aequetur functioni homogeneae duarum dimensionum ipsarum x et p, resolutio succedet. Quodsi enim fuerit $\partial y = P\,\partial x + Q\,\partial p$, aequatio solutionem continens $(P - p)\,\partial x + Q\,\partial p = 0$, erit homogenea, fietque per se integrabilis, si diuidatur per $(P - p)\,x + Q\,p$.

Corollarium 1.

696. Pro casu quarto si ponatur $y = zz$, aequatio proposita debet esse homogenea inter tres variabiles x, z et p. Vnde si proponatur aequatio homogenea quaecunque inter x, z et p, in qua hae ternae litterae x, z et p vbique eundem dimensionum numerum constituant, problema semper resolutionem admittit.

Corollarium 2.

697. Simili modo conuersis variabilibus, si ponatur $x = vv$ et $\frac{\partial x}{\partial y} = q$, vt sit $p = \frac{1}{q}$; ac proponatur aequatio homogenea quaecunque inter y, v et q, problema itidem resolui potest.

Scholion.

698. Pro casu quarto, vt aequatio $(P - p)\,\partial x + Q\,\partial p = 0$ fiat homogenea, conditiones magis amplificari possunt. Ponatur enim $x = v^{\mu}$ et $p = q^{\nu}$, sitque facta substitutione haec

aequatio

$$\mu (P - q^\nu) v^{\mu - 1} \partial v + \nu Q q^{\nu - 1} \partial q = 0$$

homogenea inter v et q, eritque P functio homogenea ν dimensionum, et Q functio homogenea μ dimensionum. Cum iam sit

$$\partial y = P \partial x + Q \partial p = \mu P v^{\mu - 1} \partial v + \nu Q q^{\nu - 1} \partial q,$$

erit y functio homogenea $\mu + \nu$ dimensionum. Quare posito $y = z^{\mu + \nu}$ problema resolutionem admittit, si inter x, y et p eiusmodi relatio proponatur, vt positio $y = z^{\mu + \nu}$, $x = v^\mu$ et $p = q^\nu$ habeatur aequatio homogenea inter ternas quantitates z, v et q, ita vt dimensionum ab iis formatarum numerus vbique sit idem. Ac si proposita fuerit huiusmodi aequatio homogenea inter z, v et q, solutio problematis ita expedietur. Cum sit $\partial y = p \partial x$, erit

$$(\mu + \nu) z^{\mu + \nu - 1} \partial z = \mu v^{\mu - 1} q^\nu \partial v;$$

ponatur iam $z = r q$ et $v = s q$, et aequatio proposita tantum duas litteras r et s continebit, ex qua alteram per alteram definire licet, tum autem per has substitutiones prodibit haec aequatio

$$(\mu + \nu) r^{\mu + \nu - 1} q^{\mu + \nu - 1} (r \partial q + q \partial r) = \mu s^{\mu - 1} q^{\mu + \nu - 1} (s \partial q + q \partial s),$$

ex qua oritur

$$\frac{\partial q}{q} = \frac{\mu s^{\mu - 1} \partial s - (\mu + \nu) r^{\mu + \nu - 1} \partial r}{(\mu + \nu) r^{\mu + \nu} - \mu s^\mu},$$

quae est aequatio differentialis separata, quoniam s per r datur. Quin etiam bini casus allati manifesto continentur in formulis $y = z^{\mu + \nu}$, $x = v^\mu$ et $p = q^\nu$; prior scilicet si $\mu = 1$ et $\nu = 1$, posterior vero si $\mu = 2$ et $\nu = -1$. Hos igitur casus perinde ac praecedentes exemplis illustrari conueniet, quorum primus praecipue est memorabilis, cum per differentia-

tiationem aequationis propositae $y = p x + \Pi$ statim praebeat aequationem integralem quaesitam, neque integratione omnino sit opus, siquidem alteram solutionem ex $\partial p = 0$ natam excludamus.

Exemplum 1.

699. *Proposita aequatione differentiali*

$$y \partial x - x \partial y = a \surd (\partial x^2 + \partial y^2)$$

eius integrale inuenire.

Posito $\frac{\partial y}{\partial x} = p$ fit $y - p x = a \surd (1 + p p)$, quae aequatio differentiata, ob $\partial y = p \partial x$, dat $- x \partial p = \frac{a p \partial p}{\surd (1 + p p)}$, quae cum sit diuisibilis per ∂p praebet primo $p = \alpha$, hincque $y = \alpha x + a \surd (1 + \alpha \alpha)$. Alter vero factor suppeditat $x = \frac{-a p}{\surd (1 + p p)}$, hincque

$$y = \frac{-a p p}{\surd (1 + p p)} + a \surd (1 + p p) = \frac{a}{\surd (1 + p p)},$$

vnde fit $x x + y y = a a$, quae est etiam aequatio integralis, sed quia nouam constantem non inuoluit, non pro completo integrali haberi potest. Integrale autem completum duas aequationes complectitur. Scilicet

$$y = \alpha x + a \surd (1 + \alpha \alpha) \text{ et } x x + y y = a a,$$

quae in hac vna comprehendi possunt

$$[(y - \alpha x)^2 - a a (1 + \alpha \alpha)] (x x + y y - a a) = 0.$$

Scholion.

700. Nisi hoc modo operatio instituatur, solutio huius quaestionis fit satis difficilis. Si enim aequationem differentialem $y \partial x - x \partial y = a \surd (\partial x^2 + \partial y^2)$ quadrando ab irrationalitate liberemus, indeque rationem $\frac{\partial y}{\partial x}$ per radicis extractionem definiamus, fit

$$(xx-aa)\partial y - xy\partial x = \pm a\partial x\sqrt{(xx+yy-aa)}$$

quae aequatio per methodos cognitas difficulter tractatur. Multiplicator quidem inueniri poteſt vtrumque membrum per ſe integrabile reddens; prius enim membrum $(xx-aa)\partial y - xy\partial x$ diuiſum per $y(xx-aa)$ fit integrabile, integrali exiſtente $= l\frac{y}{\sqrt{(xx-aa)}}$: vnde in genere multiplicator id integrabile reddens eſt

$$\frac{1}{y(xx-aa)}\Phi : \frac{y}{\sqrt{(xx-aa)}}$$

quae functio ita determinari debet, vt eodem multiplicatore quoque alterum membrum $a\partial x\sqrt{(xx+yy-aa)}$ fiat integrabile. Talis autem multiplicator eſt:

$$\frac{1}{y(xx-aa)}\cdot\frac{y}{\sqrt{(xx+yy-aa)}} = \frac{1}{(xx-aa)\sqrt{(xx+yy-aa)}}$$

quo fit

$$\frac{(xx-aa)\partial y - xy\partial x}{(xx-aa)\sqrt{(xx+yy-aa)}} = \frac{\pm a\partial x}{xx-aa}.$$

Iam ad integrale prioris membri inueſtigandum, ſpectetur x vt conſtans, eritque integrale

$$= l[y+\sqrt{(xx+yy-aa)}] + X,$$

denotante X functionem quampiam ipſius x, ita comparatam, vt ſumta iam y conſtante fiat

$$\frac{x\partial x}{[y+\sqrt{(xx+yy-aa)}]\sqrt{(xx+yy-aa)}} + \partial X = \frac{-xy\partial x}{(xx-aa)\sqrt{(xx+yy-aa)}}$$

ſeu

$$\frac{-x\partial x[y-\sqrt{(xx+yy-aa)}]}{(xx-aa)\sqrt{(xx+yy-aa)}} + \partial X = \frac{-xy\partial x}{(xx-aa)\sqrt{(xx+yy-aa)}},$$

vnde fit

$$\partial X = \frac{-x\partial x}{xx-aa} \text{ et } X = l\frac{C}{\sqrt{(xx-aa)}}.$$

Quare integrale quaeſitum eſt

$$l[y+\sqrt{(xx+yy-aa)}] + l\frac{C}{\sqrt{(xx-aa)}} = \pm\tfrac{1}{2}l\frac{a+x}{a-x},$$

vnde fit

$y +$

$$y + \sqrt{(xx + yy - aa)} = \alpha (x \pm a), \text{ hincque}$$
$$xx - aa = \alpha\alpha (x \pm a)^2 - 2\alpha (x \pm a) y, \text{ vel}$$
$$x \mp a = \alpha\alpha (x \pm a) - 2\alpha y$$

quae autem tantum eſt altera binarum aequationum integralium, altera autem aequatio integralis $xx + yy = aa$ iam quaſi per diuiſionem de calculo ſublata eſt cenſenda. Caeterum eadem ſolutio aequationis

$$(aa - xx)\partial y + xy\partial x = \pm a\partial x \sqrt{(xx + yy - aa)}$$

facilius inſtituitur ponendo $y = u\sqrt{(aa - xx)}$, vnde fit

$$(aa - xx)^{\frac{3}{2}} \partial u = \pm a\partial x \sqrt{(aa - xx)(uu - 1)} \text{ ſeu}$$
$$\frac{\partial u}{\sqrt{(uu - 1)}} = \frac{\pm a\partial x}{aa - xx},$$

cui quidem ſatisfit ſumendo $u = 1$, neque tamen hic caſus in aequatione integrali continetur, vti ſupra iam oſtendimus. Ex quo ſuſpicari liceret alteram ſolutionem $xx + yy = aa$ adeo eſſe excludendam, quod tamen ſecus ſe habere deprehenditur; ſi ipſam aequationem primariam $\frac{y\partial x - x\partial y}{\sqrt{(\partial x^2 + \partial y^2)}} = \alpha$ perpendamus. Si enim x et y ſint coordinatae rectangulae lineae curuae, formula $\frac{y\partial x - x\partial y}{\sqrt{(\partial x^2 + \partial y^2)}}$ exprimit perpendiculum ex origine coordinatarum in tangentem dimiſſum, quod ergo conſtans eſſe debet. Hoc autem euenire in circulo, origine in centro conſtituta, dum aequatio fit $xx + yy = aa$, per ſe eſt manifeſtum. Atque hinc realitas harum ſolutionum, quae minus congruae videri poterant, confirmatur, etiamſi earum ratio haud ſatis clare perſpicitur.

Exemplum 2.

701. *Propoſita aequatione differentiali*

$$y\partial x - x\partial y = \frac{a(\partial x^2 + \partial y^2)}{\partial x}$$

eius integrale inuenire.

 Poſito

Posito $\partial y = p\,\partial x$, fit $y - p x = a(1 + pp)$, et differentiando $-x\,\partial p = 2ap\,\partial p$; vnde concluditur vel $\partial p = 0$, et $p = \alpha$, hincque $y = \alpha x + a(1 + \alpha\alpha)$, vel $x = -2ap$ et $y = a(1 - pp)$, sicque, ob $p = \frac{-x}{2a}$, habebitur $4ay = 4aa - xx$, quae aequatio ad geometriam translata illam conditionem omnino adimplet.

Ex aequatione autem proposita radicem extrahendo reperitur

$$2a\,\partial y + x\,\partial x = \partial x\sqrt{(xx + 4ay - 4aa)},$$

quae posito $y = u(4aa - xx)$, abit in

$$2a\,\partial u(4aa - xx) - x\,\partial x(4au - 1)$$
$$= \partial x\sqrt{(4aa - xx)(4au - 1)},$$

haecque posito $4au - 1 = tt$, in

$$t\,\partial t(4aa - xx) - ttx\,\partial x = t\,\partial x\sqrt{(4aa - xx)},$$

quae cum sit diuisibilis per t, concludere licet $t = 0$, ideo $u = \frac{1}{4a}$, atque hinc $4ay = 4aa - xx$.

Exemplum 3.

702. *Proposita aequatione differentiali*

$$y\,\partial x - x\,\partial y = a\sqrt[3]{(\partial x^3 + \partial y^3)},$$

eius integrale assignare.

Haec aequatio more consueto, si rationem $\frac{\partial y}{\partial x}$ inde extrahere vellemus, vix tractari posset. Posito autem $\partial y = p\,\partial x$ fit $y - px = a\sqrt[3]{(1 + p^3)}$, et differentiando $x\,\partial p = \frac{-app\,\partial p}{\sqrt[3]{(1 + p^3)^2}}$, vnde duplex conclusio deducitur, vel $\partial p = 0$ et $p = \alpha$, sicque $y = \alpha x + a\sqrt[3]{(1 + \alpha^3)}$, vel

$x =$

$$x = \frac{-a p p}{\sqrt[3]{(1+p^3)^2}} \text{ et } y = \frac{a}{\sqrt[3]{(1+p^3)^2}},$$

vnde fit $p p = -\frac{x}{y}$, et ob $y^3 (1+p^3)^2 = a^3$, erit $p^3 = \frac{a\sqrt{a}}{y\sqrt{y}} - 1$, hincque $\frac{(a\sqrt{a} - y\sqrt{y})^2}{y^3} = -\frac{x^3}{y^3}$, seu $x^3 + (a\sqrt{a} - y\sqrt{y})^2 = 0$.

Exemplum 4.

703. *Proposita aequatione differentiali*

$$y \partial x - n x \partial y = a \sqrt{(\partial x^2 + \partial y^2)},$$

eius integrale inuenire.

Posito $\partial y = p \partial x$, habetur $y - n p x = a \sqrt{(1 + p p)}$, vnde differentiando elicitur

$$(1-n) p \partial x - n x \partial p = \frac{a p \partial p}{\sqrt{(1+pp)}}, \text{ siue}$$

$$\partial x - \frac{n x \partial p}{(1-n) p} = \frac{a \partial p}{(1-n)\sqrt{(1+pp)}},$$

quae per $p^{\frac{n}{n-1}}$ multiplicata et integrata praebet

$$p^{\frac{n}{n-1}} x = \frac{a}{1-n} \int \frac{p^{\frac{n}{n-1}} \partial p}{\sqrt{(1+pp)}}.$$

Hinc deducimus casus sequentes, integrationem admittentes

si $n = \frac{3}{2}$; $p^3 x = C - \frac{2}{3} a (p p - \frac{2}{1}) \sqrt{(1+pp)}$,

si $n = \frac{5}{4}$; $p^5 x = C - \frac{4}{5} a (p^4 - \frac{4}{3} p^2 + \frac{4.2}{3.1}) \sqrt{(1+pp)}$,

si $n = \frac{7}{6}$; $p^7 x = C - \frac{6}{7} a (p^6 - \frac{6}{5} p^4 + \frac{6.4}{5.3} p^2 - \frac{6.4.2}{5.3.1}) \sqrt{(1+pp)}$,

ac si $n = \frac{2\lambda+1}{2\lambda}$, erit $y = p x + a \sqrt{(1+pp)} + \frac{px}{2\lambda}$, et

$$x = \frac{C}{p^{2\lambda+1}} - \frac{2\lambda a}{(2\lambda+1)p}\left(1 - \frac{2\lambda}{(2\lambda-1)pp} + \frac{2\lambda(2\lambda-2)}{(2\lambda-1)(2\lambda-3)p^4} - \text{etc.}\right)\sqrt{(1+pp)}.$$

Quodsi ergo sumatur $\lambda = \infty$, vt sit $n = 1$, erit

$y =$

$$y = p x + a \surd (1 + p p), \text{ et } x = \frac{C}{p^{2\lambda+1}} - \frac{a p}{\surd (1 + p p)},$$

vnde ſi conſtans C ſit $= 0$, ſtatim ſequitur ſolutio ſuperior $x x + y y = a a$. At ſi conſtans C non euaneſcat, minimum diſcrimen in quantitate p infinitam varietatem ipſi x inducit. Quantumuis ergo x varietur, quantitas p vt conſtans ſpectari poteſt, vnde poſito $p = \alpha$, altera ſolutio $y = \alpha x + a \surd (1 + \alpha \alpha)$ obtinetur. Hinc ergo dubium ſupra, circa exemplum 1. natum, non mediocriter illuſtratur.

Exemplum 5.

704. *Propoſita aequatione differentiali*

$$A \partial y^n = (B x^\alpha + C y^\beta) \partial x^n$$

exiſtente $n = \frac{\alpha \beta}{\alpha - \beta}$, *eius integrale inueſtigare.*

Poſito $\frac{\partial y}{\partial x} = p$ erit $A p^n = B x^\alpha + C y^\beta$. Ponamus iam $p = q^{\alpha\beta}$, $x = v^{\beta n}$ et $y = z^{\alpha n}$, vt habeamus hanc aequationem homogeneam $A q^{\alpha\beta n} = B v^{\alpha\beta n} + C z^{\alpha\beta n}$, quae poſitis $z = r q$ et $v = s q$, abit in $A = B s^{\alpha\beta n} + C r^{\alpha\beta n}$. Cum vero ſit

$$\partial y = \alpha n z^{\alpha n - 1} \partial z = \alpha n r^{\alpha n - 1} q^{\alpha n - 1} (r \partial q + q \partial r) \text{ et}$$

$$p \partial x = \beta n v^{\beta n - 1} q^{\alpha\beta} \partial v = \beta n s^{\beta n - 1} q^{\alpha\beta + \beta n - 1} (s \partial q + q \partial s),$$

erit

$$\alpha r^{\alpha n - 1} (r \partial q + q \partial r) = \beta s^{\beta n - 1} q^{\alpha\beta + \beta n - \alpha n} (s \partial q + q \partial s).$$

Eſt vero per hypotheſin $\alpha\beta + \beta n - \alpha n = 0$, vnde oritur

$$\alpha r^{\alpha n} \partial q + \alpha r^{\alpha n - 1} q \partial r = \beta s^{\beta n} \partial q + \beta s^{\beta n - 1} q \partial s,$$

hincque

$$\frac{\partial q}{q} = \frac{\alpha r^{\alpha n - 1} \partial r - \beta s^{\beta n - 1} \partial s}{\beta s^{\beta n} - \alpha r^{\alpha n}}.$$

At eſt

$s^{\beta n} =$

$$s^{\beta n} = \left(\frac{A - C\, r^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}}, \text{ hincque}$$

$$\beta\, s^{\beta n - 1}\, \partial s = -\frac{\beta C}{B}\, r^{\alpha\beta n - 1}\, \partial r \left(\frac{A - C\, r^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}},$$

vnde fit

$$\frac{\partial q}{q} = \frac{\alpha\, r^{\alpha n - 1}\, \partial r + \frac{\beta C}{B}\, r^{\alpha\beta n - 1}\, \partial r \left(\frac{A - C\, r^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}}}{\beta \left(\frac{A - C\, r^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}} - \alpha\, r^{\alpha n}}.$$

Facilius autem calculus hoc modo inſtituetur; ſumto $A = 1$, erit

$$p = \tfrac{\partial y}{\partial x} = (B\, x^{\alpha} + C\, y^{\beta})^{\frac{1}{n}},$$

ſit $y = x^{\frac{\alpha}{\beta}}\, u$, fiet

$$x^{\frac{\alpha}{\beta}}\, \partial u + \tfrac{\alpha}{\beta}\, x^{\frac{\alpha - \beta}{\beta}}\, u\, \partial x = x^{\frac{\alpha}{n}}\, \partial x\, (B + C\, u^{\beta})^{\frac{1}{n}},$$

quae aequatio, cum ſit $\frac{\alpha}{n} = \frac{\alpha - \beta}{\beta}$, abit in hanc

$$\beta\, x\, \partial u + \alpha\, u\, \partial x = \beta\, \partial x\, (B + C\, u^{\beta})^{\frac{1}{n}},$$

vnde fit

$$\frac{\partial x}{x} = \frac{\beta\, \partial u}{\beta\, (B + C\, u^{\beta})^{\frac{1}{n}} - \alpha\, u},$$

ſicque x per u determinatur, et quia $u = x^{-\frac{\alpha}{\beta}}\, y$, habebitur aequatio inter x et y.

Scholion.

705. Hoc igitur modo operationem inſtitui conueniet, quando inter binas variabiles x et y vna cum differentialium

 ratione

ratione $\frac{\partial y}{\partial x} = p$, eiusmodi relatio proponitur, ex qua valor ipsius p commode elici non poteſt. Tum ergo calculum ita tractari oportet, vt per differentiationem ponendo $\partial y = p \partial x$ vel $\partial x = \frac{\partial y}{p}$, tandem perueniatur ad aequationem differentialem ſimplicem inter duas tantum variabiles, quem in finem etiam ſaepe idoneis ſubſtitutionibus vti neceſſe eſt. Atque hucusque fere Geometris in reſolutione aequationum differentialium primi gradus etiamnum pertingere licuit, vix enim vlla via integralia inueſtigandi adhuc quidem adhibita hic praetermiſſa videtur. Num autem multo maiorem calculi integralis promotionem ſperare liceat? vix equidem affirmauerim, cum plurima extent inuenta, quae ante vires ingenii humani ſuperare videbantur.

Cum igitur calculum integralem in duos libros ſim partitus, quorum prior circa relationem binarum tantum variabilium, poſterior vero ternarum pluriumue verſatur, atque iam libri primi partem priorem in differentialibus primi ordinis conſtitutam hic pro viribus expoſuerim, ad eius alteram partem progredior, in qua binarum variabilium relatio ex data differentialium ſecundi altiorisue ordinis conditione requiritur.